# COLLEGE ALGEBRA

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# A BRIEF COURSE
IN
# COLLEGE ALGEBRA

BY
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# PREFACE TO THE FIRST EDITION

THE present book, which is intended as a text for use in the freshman year of college or technical school, has been prepared with the following considerations particularly in mind.

(1) In view of the fact that a considerable number of pupils enter college today for whom no knowledge of quadratic equations can be assumed, it seems desirable to include a complete treatment of this important topic in the present-day college text. Two chapters (II and III) are therefore devoted to quadratic equations, the first of which, however, is altogether elementary and may be omitted at the discretion of the teacher.

(2) In order to meet the needs and customs of different institutions, the various chapters have been made quite independent of each other, thus permitting a ready adjustment of the book to either a long course or a short one, and affording the teacher the greatest possible flexibility in the choice of topics for any course of given length. In this connection the author feels that it should be frankly recognized that today college algebra in most institutions is pursued but a few weeks. This makes it impossible to cover a wide range of topics and forces such a selection as may fit best the needs of the particular situation. Much may be gained, however, from a brief but intensive study of a few special topics in algebra at this period of the pupil's career.

(3) In view of the importance in elementary physics and other applied fields of the subject of variation, this topic has been treated somewhat more fully than usual. On the other hand, such topics as complex numbers (vector addition, multiplication, etc.), partial fractions, limits and infinite series

have been omitted in the belief that, even in case there is time to include them in the course, they may be taken up to greater advantage at a later time when the pupil is face to face with their chief applications.

(4) The ideal problem for a freshman text is a short one which illustrates pointedly the mathematical principle at issue. Problems long in statement and dealing with the technique of the arts and sciences should have but little place in the freshman year. At this period the essential task of both teacher and text should be to train the pupil in accuracy and conciseness of thought and expression, the mathematics itself forming, for the most part, the medium through which this may be accomplished.

Certain sections of the book have been starred (*) to indicate that they are of minor importance. These may be omitted without destroying the continuity of the whole.

The author would here express his thanks to Professor E. B. Lytle, of the University of Illinois, who read the manuscript and offered valuable suggestions, and to Professor J. L. Markley and Mr. R. W. Barnard, of the University of Michigan, who assisted in reading the proofs.

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## PREFACE TO THE REVISED EDITION

Experience having shown that a treatment of complex numbers is desired in certain institutions, a chapter upon this subject has been added (Chap. XV); also a collection of 167 miscellaneous exercises from elementary algebra (pages 245–250), these affording abundant material for a review of high school algebra at the beginning of the course in college algebra.

The author would take this opportunity to thank all those who have contributed to the large measure of success enjoyed by the first edition of the book.

W. B. F.

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# COLLEGE ALGEBRA

## CHAPTER I

## REVIEW TOPICS

**1. Algebraic Reductions.** The process of reducing, or simplifying, a given algebraic expression makes frequent use of the following principles from elementary algebra.†

PRINCIPLE 1. *A parenthesis preceded by a minus sign may be removed from an expression if the signs of all the terms in the parenthesis are changed.*

Thus $a-(b-c)=a-b+c.$

Likewise $a+b-(c-d+e)=a+b-c+d-e.$

A parenthesis preceded by a *plus* sign may be removed without changes in the signs of the terms which it includes.

Thus $a+(b-c)=a+b-c.$

Likewise $a+b+(c-d+e)=a+b+c-d+e.$

### EXERCISES

Simplify each of the following expressions by removing all parentheses and combining terms wherever possible.

**1.** $x-(y-z).$

**2.** $x-(-y-z).$

**3.** $-(a+b)+2.$

**4.** $m-(n-2a).$

**5.** $5a-2b-(a-2b).$ *Ans.* $4a.$

**6.** $a-(b-c+a)-(c-b).$

---

† This chapter treats not all topics of elementary algebra but only a few topics that are important for the work which follows. The student should have for reference a textbook in elementary algebra, preferably the one he used in high school. Attention is called, however, to the extended list of exercises from high school algebra to be found on pages 245–250 of the present text.

7. $2xy+3y^2-(x^2+xy-y^2)$.
8. $m+(3m-n)-(2n-m)+n$.
9. $a^2b+b^2c+a^2c^2-(2a^2b^2-3a^2c)+(4a^2b-5a^2c^2-6a^2b^2)$.
10. $\dfrac{x+(y-z)-(x-y)}{a+b-(2a+b-c)}$.
11. $(a+b)^2-(a-b)^2$.
12. $\dfrac{2ab-(a+b)^2}{x^2-(x-y)^2}$.
13. $a(b-c)+b(a-c)-c(a-b)$.

PRINCIPLE 2. *Multiplying or dividing both the numerator and the denominator of a fraction by the same number does not change the value of the fraction.*

Thus

$$\frac{2}{3}=\frac{2\times2}{3\times2}=\frac{4}{6}.$$

Likewise

$$\frac{8}{10}=\frac{8\div2}{10\div2}=\frac{4}{5}.$$

Also

$$\frac{a}{b}=\frac{a\times a}{b\times a}=\frac{a^2}{ab};\qquad \frac{m^2n}{mn^2}=\frac{m^2n\div mn}{mn^2\div mn}=\frac{m}{n}.$$

This principle is frequently used to change, or reduce a fraction to a form having a given denominator.

Thus, suppose it is desired to change the fraction

$$a/(a+b)$$

to a form having $a^2-b^2$ as its denominator. To do so, we multiply both numerator and denominator by $a-b$, as follows:

$$\frac{a}{a+b}=\frac{a(a-b)}{(a+b)(a-b)}=\frac{a^2-ab}{a^2-b^2}.\quad Ans.$$

The principle is also used to reduce a fraction to its lowest terms.

Thus, suppose we are to reduce the fraction

$$\frac{21a^2x^2y}{30a^3xz}$$

to its lowest terms. The process consists in dividing both numerator and denominator by all the factors which they have in common; that is, in the present case, by 3, by $a^2$, and by $x$. In practice, the work is done by *cancellation* as shown below:

$$\frac{\overset{7\qquad x}{\cancel{21}\cancel{a^2}\cancel{x^2}y}}{\underset{10a}{\cancel{30}\cancel{a^3}\cancel{x}z}} = \frac{7xy}{10az}.$$

## EXERCISES

**1.** Change 2/3 to a fraction whose denominator is 21.

**2.** Change 4/5 to a fraction whose denominator is 125.

**3.** Change $5a/7$ to a fraction whose denominator is 42.

**4.** Change $\frac{4a^2}{5y}$ to a fraction whose denominator is $20y^3$.

**5.** Change $\frac{x-3}{x-1}$ to a fraction whose denominator is $(x-1)^2$.

**6.** Reduce $\frac{a}{3-a}$ to a fraction whose denominator is $9-a^2$.

**7.** Reduce $\frac{d-c}{b-a}$ to a fraction whose denominator is $a-b$.

Reduce each of the following fractions to its lowest terms.

**8.** $\frac{a^2xy^2}{a^3xy}$.

**9.** $\frac{a^2b^2x^2}{b^3xy^2}$.

**10.** $\frac{16m^2nx^2z^2}{40am^3yz^3}$.

**11.** $\frac{77a^7x^5b^3y}{121a^3b^5c^7}$.

**12.** $\frac{x^{m+1}y}{xy^{m+1}}$.

**13.** $\frac{a^2-b^2}{(a+b)^2}$.

**14.** $\frac{a^2-2ab+b^2}{a^2-b^2}$.

**15.** $\frac{3a^2+3ab}{a^3b-ab^3}$.

**16.** $\frac{3a^2b-3b^3}{2ab-2b^2}$.

**17.** $\frac{a^4bc-b^5c}{3a^2b+3b^3}$.

**18.** $\frac{a(a+2b)^4}{b(a^2-4b^2)^2}$.

**19.** $\frac{x^2-2x^4+x^6}{x^2-x^6}$.

**20.** $\frac{m-m^2-n+mn}{m-mn+n^2-n}$.

**21.** $\frac{ab+ac-db-dc}{mb+mc}$.

**2. Addition and Subtraction of Fractions.** In case two fractions have the same denominator, their sum will be equal to the sum of their numerators divided by this denominator.

Thus $$\frac{2}{3}+\frac{5}{3}=\frac{2+5}{3}=\frac{7}{3}.$$

Likewise $$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b};$$

$$\frac{mn}{x^2y}+\frac{p-q}{x^2y}=\frac{mn+p-q}{x^2y}.$$

In case two fractions do *not* have the same denominator, they may be added by first changing them, as in §1, so that they shall come to have equal denominators, and then proceeding as mentioned above.

Thus $$\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{8+9}{12}=\frac{17}{12}=1\tfrac{5}{12}.$$

Likewise $$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd};$$

$$\frac{m+n}{m-n}+\frac{m-n}{m+n}=\frac{(m+n)^2}{m^2-n^2}+\frac{(m-n)^2}{m^2-n^2}=\frac{m^2+2mn+n^2+m^2-2mn+n^2}{m^2-n^2}=\frac{2m^2+2n^2}{m^2-n^2}.$$

In practice, when adding several fractions, it is best to determine first the ***least common multiple*** (***L. C. M.***) of the several denominators, that is, the expression of lowest degree which exactly contains each of them, then change each fraction so that its denominator shall be this L. C. M. and add as indicated above.

Thus, in adding 2/15 and 3/10, the L. C. M. of the denominators is 30. The two fractions, when changed so as to have 30 as denominator, are respectively 4/30 and 9/30. Hence the desired sum is (4+9)/30=13/30.

Likewise, in adding $a/(m^2n)$ and $b/(mn^2)$, the L. C. M. of the denominators is $m^2n^2$, so that

$$\frac{a}{m^2n}+\frac{b}{mn^2}=\frac{an}{m^2n^2}+\frac{bm}{m^2n^2}=\frac{an+bm}{m^2n^2}. \quad \textit{Ans.}$$

Similar statements apply whenever one fraction is to be subtracted from another, or when both addition and subtraction are involved any number of times.

Thus

$$\frac{3a}{5}+\frac{b}{2}-3+\frac{1}{b}=\frac{6ab}{10b}+\frac{5b^2}{10b}-\frac{30b}{10b}+\frac{10}{10b}=\frac{6ab+5b^2-30b+10}{10b}.\quad \textit{Ans.}$$

Likewise

$$\frac{a-b}{a+b}-\frac{a+b}{a-b}+\frac{6ab}{a^2-b^2}=\frac{(a-b)^2}{a^2-b^2}-\frac{(a+b)^2}{a^2-b^2}+\frac{6ab}{a^2-b^2}=$$

$$\frac{(a-b)^2-(a+b)^2+6ab}{a^2-b^2}=\frac{a^2-2ab+b^2-a^2-2ab-b^2+6ab}{a^2-b^2}=\frac{2ab}{a^2-b^2}.\quad \textit{Ans.}$$

## EXERCISES

Simplify each of the following expressions by performing the indicated additions and subtractions.

**1.** $\frac{3x}{4}+\frac{7x}{10}$.

**2.** $\frac{5x-1}{8}-\frac{3x-2}{7}+\frac{x-5}{4}$.

**3.** $\frac{a-b}{ab}+\frac{b-c}{bc}$.

**4.** $\frac{ax}{a-x}+a$.

Hint. $a=\frac{a}{1}=\frac{a(a-x)}{a-x}$.

**5.** $a+b+\frac{a^2+b^2}{a-b}$.

**6.** $\frac{b-c}{bc}-\frac{a-c}{ac}$.

**7.** $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

**8.** $x+y-\frac{x^2+y^2}{x-y}$.

**9.** $\frac{x}{x-2}-\frac{x-2}{x+2}$.

**10.** $\frac{x-4}{3}-\frac{x-6}{8}+2-\frac{x+8}{6}$.

**11.** $1-\frac{ax-bx+ab}{x^2}$.

**12.** $\frac{a+1}{a^2-9}-\frac{6}{a+5}+\frac{10}{a+3}$.

**13.** $m-\frac{m^2+n^2}{m-n}+n$.

**14.** $\frac{a}{a-2}-\frac{a-2}{a+2}+\frac{3}{4-a^2}$.

Hint. $\frac{3}{4-a^2}=\frac{-3}{a^2-4}$.

**15.** $\frac{a+b}{a-b}-\frac{a^2+b^2}{b^2-a^2}+\frac{b-a}{a+b}$.

**16.** $\frac{(a+b)^2}{a^2+b^2}-1+\frac{2ab}{a^2-b^2}$.

**17.** $\frac{1}{x}+1+\frac{2x}{1+x}-2$.

### 3. Multiplication and Division of Fractions.

**Principle 1.** *In order to multiply two or more fractions together, multiply their numerators together to get the numerator of the product, and multiply their denominators together to get the denominator of the product.*

In performing such multiplications, it is desirable to cancel like *factors* from numerator and denominator wherever possible.

Thus

$$\frac{1}{2}\times\frac{2}{3}\times\frac{6}{7}=\frac{1\times\cancel{2}\times\overset{2}{\cancel{6}}}{\cancel{2}\times\cancel{3}\times 7}=\frac{2}{7}.$$

Likewise

$$\frac{3ab}{4xy}\times\frac{2y}{3a^2}=\frac{\cancel{3}\cancel{a}b\cdot\cancel{2}\cancel{y}}{\underset{2}{\cancel{4}}x\cancel{y}\cdot\cancel{3}\underset{a}{\cancel{a^2}}}=\frac{b}{2ax}.$$

Similarly

$$\frac{a-b}{a+b}\times\frac{d}{a^2-b^2}=\frac{\cancel{(a-b)}d}{(a+b)\cancel{(a-b)}(a+b)}=\frac{d}{(a+b)^2}.$$

**Principle 2.** *In order to divide one fraction by another, invert the divisor and proceed as in multiplication.*

Thus

$$\frac{1}{2}\div\frac{2}{3}=\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}.$$

Likewise

$$\frac{5mn}{6bx}\div\frac{10m^2n}{3ax^2}=\frac{\cancel{5}\cancel{m}\cancel{n}}{\underset{2}{\cancel{6}}bx}\times\frac{\cancel{3}a\overset{x}{\cancel{x^2}}}{\underset{2m}{\cancel{10m^2n}}}=\frac{ax}{4bm}.$$

Similarly

$$\frac{(a-b)^2}{a+b}\div\frac{a^2-ab}{b}=\frac{\overset{a-b}{\cancel{(a-b)^2}}}{a+b}\times\frac{b}{a\cancel{(a-b)}}=\frac{b(a-b)}{a(a+b)}.$$

EXERCISES

Perform each of the following indicated multiplications.

1. $\frac{5xy}{2ac}\times\frac{3ax}{10y^2}$.

2. $\frac{4mn}{3xy}\times-\frac{15bx}{16m^2}$.

3. $\frac{2ax}{12by}\times\frac{10b^2}{x^2}$.

4. $\frac{a^mb^n}{4c}\times\frac{6c^2}{a^{m-1}b^{2n}}$.

**5.** $\dfrac{a}{a+b}\times\dfrac{b}{a-b}.$

**6.** $\dfrac{xy^2}{20-8x}\times\dfrac{25-10x}{x^2y}.$

**7.** $\dfrac{(a-b)^2}{a+b}\times\dfrac{b}{a^2-ab}\times\dfrac{(a+b)^2}{a^2-b^2}.$

**8.** $\dfrac{4a-b}{2x+y}\times\dfrac{2a}{4a^2-ab}\times\dfrac{4x^2-y^2}{4}.$

**9.** $\dfrac{x^2+5x+6}{x^2+6x+5}\times\dfrac{x^2+7x+10}{x^2+7x+12}.$

**10.** $\left(1+\dfrac{2}{m-1}\right)\left(\dfrac{m^2+m-2}{m^2+m}\right).$

Perform each of the following indicated divisions.

**11.** $\dfrac{12a^4b}{25ac}\div\dfrac{4ax}{24c^2}.$

**12.** $\dfrac{5ab}{3a^2c^2}\div\dfrac{25b^2}{15a^2}.$

**13.** $\dfrac{7x^3}{4y^3}\div\dfrac{21x^2y^2}{14a}.$

**14.** $\dfrac{my-y^2}{(m+y)^2}\div\dfrac{y^2}{m^2-y^2}.$

**15.** $(4a+2)\div\dfrac{2a+1}{5a}.$

**16.** $\dfrac{x^2-y^2}{x+2y}\div(x^2-3xy+2y^2).$

**17.** $\dfrac{x^2+x-2}{x^2-5x+4}\div\dfrac{x^2-x-6}{x^2+x-20}.$

**18.** $\left(x\div\dfrac{1}{y}\right)\div\left(y^2\div\dfrac{1}{x^2}\right).$

**19.** $\left(\dfrac{a^3}{b}\div b^2\right)\div\left(\dfrac{a^2}{b^2}\times ab\right).$

**20.** $(a+c)\div\left(\dfrac{a^2-c^2}{1+x}\div\dfrac{a-c}{1-x^2}\right).$

Simplify each of the following expressions.

**21.** $\left(y-x+\dfrac{x^2}{y}\right)\div\left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right).$

**22.** $\left(x+\dfrac{3x+6}{x^2-1}+2\right)\div\left(x+3+\dfrac{1}{x+1}\right).$

**23.** $\left(1+\dfrac{1}{y^2}+\dfrac{1}{y^4}\right)\div\left(1+\dfrac{1}{y}+\dfrac{1}{y^2}\right).$

**24.** $\dfrac{\dfrac{x+y}{ab}}{\dfrac{x^2-y^2}{ab^2}}.$ **25.** $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}.$ **26.** $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}.$ **27.** $\dfrac{\dfrac{x+y}{y}-\dfrac{x+y}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}.$

**4. Simple Equations.** By a *simple equation* is meant one which, when cleared of fractions, contains the unknown number to no higher power than the first. The usual method of solving such equations is illustrated below.

Example. Solve the equation

$$\frac{x+1}{2}-\frac{2x-5}{5}=\frac{11x+5}{10}-\frac{x-13}{3}.$$

Solution. The L. C. M. of the denominators is 30. Hence, multiplying both sides of the equation by 30 in order to clear of the fractions, we obtain

$$15(x+1)-6(2x-5)=3(11x+5)-10(x-13),$$

or

$$15x+15-12x+30=33x+15-10x+130.$$

Transposing and collecting like terms now gives

$$-20x=100.$$

Therefore

$$x=-5. \quad \textit{Ans.}$$

Check. Placing $x=-5$ in the original equation gives

$$\frac{-5+1}{2}-\frac{-10-5}{5}=\frac{-55+5}{10}-\frac{-5-13}{3},$$

or

$$\frac{-4}{2}+\frac{15}{5}=\frac{-50}{10}+\frac{18}{3},$$

or

$$-2+3=-5+6,$$

or

$$1=1.$$

## EXERCISES

Solve each of the following equations for $x$, checking your answer for each of the first five.

1. $\frac{2x-3}{4}+\frac{x+1}{6}=\frac{5x+2}{12}.$

2. $\frac{x-5}{3}-\frac{2x+3}{6}=1-\frac{7x+3}{12}.$

3. $\frac{1}{2x}-\frac{3}{x}+\frac{5}{3x}=\frac{3}{4x}-\frac{19}{24}.$

4. $\frac{x}{3}-\frac{x^2-5x}{3x-7}=\frac{2}{3}.$

**5.** $\frac{6x+3}{15}-\frac{3x-1}{5x-25}=\frac{2x-9}{5}$.

**6.** $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}$.

In each of the following problems, let $x$ represent the unknown quantity, then form an equation and solve it:

**7.** Divide 38 into two parts whose quotient is 7/12.

**8.** Divide 96 into two parts such that 3/4 of the greater shall exceed 3/4 of the smaller by 6.

**9.** I have $110 in one bank and $75 in another one. If I have $45 more to deposit, how shall I divide it between the two banks in order that they may have equal amounts?

**10.** A motor boat traveling at the rate of 12 miles per hour crossed a lake in 10 minutes less time than when traveling at the rate of 10 miles per hour. What was the width of the lake?

[HINT. *Time* = *Distance* ÷ *Rate*.]

**11.** A freight train goes 6 miles an hour less than a passenger train. If it goes 80 miles in the same time that a passenger train goes 112 miles, find the rate of each.

**12.** A tank can be filled by one pipe in 10 hours, or by another pipe in 15 hours. How long will it take to fill it if *both* pipes are used at the same time?

[HINT. Let $x$ = the number of hours. Then $1/x$ = the part both can fill in one hour.]

**13.** Two pipes are connected with a tank. The larger one is an intake pipe which can fill the tank in 3 hours, while the smaller one is an outlet pipe entering at the bottom which can empty the tank in 4 hours. With both pipes open, how long before the tank will fill?

**14.** A can do a piece of work in 16 hours, while B can do it in 20 hours. If A works 10 hours, how many hours must B work to finish?

**15.** An aviator made a trip of 85 miles. After flying 40 miles, he increased his speed by 15 miles an hour and made the remaining distance in the same time it took him to fly the first 40 miles. What was his rate over the first 40 miles?

**16.** A 5-gallon mixture of alcohol and water contains 80 % alcohol. How much water must be added to make it contain only 50 % alcohol?

**17.** What weight of water must be added to 65 pounds of a 10 % salt solution to reduce it to an 8 % solution?

**18.** A train 660 feet long, running at 15 miles an hour, will pass completely through a certain tunnel in 49½ minutes. How long is the tunnel?

**5. Elimination.** In case two simple equations (see § 4) are given, each containing the *two* unknown values $x$ and $y$, these values may usually be obtained by the process of ***elimination*** as is illustrated below.

EXAMPLE 1. Solve the equations

(1) $$2x+3y=2,$$

(2) $$5x-4y=28.$$

SOLUTION. From (1) we have

(3) $$2x=2-3y.$$

Therefore
$$x=\frac{2-3y}{2}.$$

Substituting this value for $x$ in (2), we find

(4) $$5\left(\frac{2-3y}{2}\right)-4y=28.$$

In (4) we have an equation containing only $y$; that is, $x$ has been *eliminated* from (1) and (2). Clearing (4) of fractions and simplifying, we obtain $-23y=46$. Therefore $y=-2$.

Substituting $-2$ for $y$ in (1), we find
$$2x-6=2, \text{ or } 2x=8, \text{ or } x=4.$$

Hence the required values of $x$ and $y$ are $x=4$ and $y=-2$.

CHECK. Substituting $x=4$ and $y=-2$ in (1), we have
$$2\times4+3(-2)=8-6=2,$$
as desired. Likewise, with $x=4$ and $y=-2$, equation (2) is satisfied, since it becomes
$$5\times4-4\times(-2)=20+8=28.$$

The preceding method of solution, wherein the value of one of the letters, as $x$, is obtained from one of the equations and then substituted in the other equation, thus giving an equation, like (4), containing only *one* letter, is called ***elimination by substitution.*** Another common and very useful method of elimination is illustrated below.

EXAMPLE 2. Solve the equations

(1) $$3x+4y=12.$$

(2) $$2x-5y=54.$$

SOLUTION. Multiplying (1) by 2 and (2) by 3, the two equations become

(3) $$6x+8y=24,$$

(4) $$6x-15y=162.$$

The coefficient of $x$ is now the same in both (3) and (4) so that, upon subtracting (4) from (3), we obtain

(5) $$23y=-138.$$

Therefore $y=-6$.

Substituting $y=-6$ in (1), we now have

$$3x-24=12, \text{ or } 3x=36.$$

Therefore $x=12$. Hence the required values of $x$ and $y$ are $x=12$, $y=-6$.

## EXERCISES

Find, by any method of elimination, the values of $x$ and $y$ in each of the following pairs of equations. Check your answers in the first five.

**1.** $\begin{cases} x-y=4, \\ 4y-x=14. \end{cases}$

**2.** $\begin{cases} 3x-4y=26, \\ x-8y=22. \end{cases}$

**3.** $\begin{cases} y+1=3x, \\ 5x+9=3y. \end{cases}$

**4.** $\begin{cases} 4y=10-x, \\ y-x=5. \end{cases}$

**5.** $\begin{cases} 1-x=3y, \\ 3(1-x)=40-y. \end{cases}$

**6.** $\begin{cases} x+\dfrac{y}{3}=11, \\ \dfrac{x}{3}+3y=21. \end{cases}$

**7.** $\begin{cases} \dfrac{x}{3}=11-\dfrac{y}{2}, \\ \dfrac{x}{3}+\dfrac{2y}{7}=8. \end{cases}$

**8.** $\begin{cases} \dfrac{3x}{4}+\dfrac{2y}{3}=20, \\ \dfrac{x}{2}+\dfrac{3y}{4}=17. \end{cases}$

**9.** $\begin{cases} \dfrac{x+y}{2}-\dfrac{x-y}{3}=8, \\ \dfrac{x+y}{3}+\dfrac{x-y}{4}=11. \end{cases}$

**10.** $\begin{cases} \dfrac{7+x}{5}-\dfrac{2x-y}{4}=3y-5, \\ \dfrac{5y-8}{2}+\dfrac{5x-3}{6}=18-5x. \end{cases}$

**11.** $\begin{cases} \dfrac{1}{x-1}-\dfrac{3}{x+y}=0, \\ \dfrac{3}{x-y}+3=0. \end{cases}$

**12.** $\begin{cases} \dfrac{4}{x}-\dfrac{3}{y}=\dfrac{11}{5}, \\ \dfrac{2}{x}+\dfrac{5}{y}=\dfrac{25}{3}. \end{cases}$

[HINT. Solve first for $1/x$ and $1/y$.]

13. $\begin{cases} \dfrac{5}{x}+\dfrac{6}{y}=64, \\ \dfrac{6}{x}+\dfrac{5}{y}=73\frac{1}{2}. \end{cases}$

14. $\begin{cases} \dfrac{5}{x}-\dfrac{3}{y}=-2, \\ \dfrac{25}{x}+\dfrac{1}{y}=6. \end{cases}$

In each of the following examples, let $x$ and $y$ represent the desired unknown quantities, form two equations and solve.

**15.** The sum of two numbers is 75 and their difference is 5. What are the numbers?

**16.** One-third the sum of two numbers is 10, while one-sixth of their difference is 1. Find the numbers.

**17.** A father's age is $1\frac{1}{2}$ that of his son. Twenty years ago his age was twice his son's. How old is each?

**18.** A part of $2500 is invested at 6% interest and the remainder at 5%. The yearly income from both is $141. Find the amount of each investment.

**19.** A and B together can do a piece of work in 12 days. After A has worked alone for 5 days, B finishes the work in 26 days. In what time could each do the work alone?

[HINT. If $x$ = the time in which A can do it alone, and $y$ = the time in which B can do it alone, then the part which A can do in one day $=1/x$, etc. See Ex. 12, p. 9 and Ex. 12, p. 11.]

**20.** An errand boy went to the bank to deposit 38 bills, some of them being $1 bills and the rest $2 bills. If their total value was $50, how many of each were there?

**21.** A grocer wishes to make 50 pounds of coffee worth 32 cents a pound by mixing two other grades, which are worth 26 and 35 cents per pound, respectively. How much of each must he use?

**22.** One cask contains 18 gallons of vinegar and 12 gallons of water; another contains 4 gallons of vinegar and 12 of water. How many gallons of each must be taken so that when mixed there may be 21 gallons containing half vinegar and half water?

**23.** Two cities are 140 miles apart. To travel the distance between them by automobile takes 3 hours less time than by bicycle, but if the bicycle has a start of 42 miles, each takes the same time. What is the rate of the automobile, and what the rate of the bicycle?

**24.** The perimeter of a certain rectangle is 16 feet. If the length be increased by 3 feet and the breadth by 2 feet, the area is increased by 25 square feet. What are the original length and breadth?

**6. Graph of an Equation.** In reviewing this topic, it is desirable first to recall the following fundamental ideas and definitions.

Let two lines $XX'$ and $YY'$ be drawn on a sheet of squared (coordinate) paper, the line $XX'$ being horizontal and $YY'$ vertical. Two such lines constitute a pair of ***coordinate axes.*** $XX'$ is called the ***x-axis,*** $YY'$ is called the ***y-axis.*** The point $O$ where they intersect is called the ***origin.***

Having chosen any point, as $P$, in the plane of the axes, draw the perpendiculars $PA$ and $PB$. Then $PA$, which is parallel to the $x$-axis, is called the ***abscissa*** of $P$, while $PB$, which is parallel to the $y$-axis, is called the ***ordinate*** of $P$. The abscissa and ordinate taken together are called the ***coordinates*** of the point $P$.

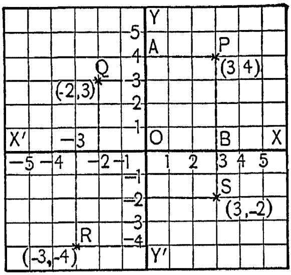

FIG. 1

Thus, in Fig. 1, the abscissa of $P$ is 3 units, while its ordinate is 4 units. Note that the $x$- and $y$- unit scales are indicated along the $x$- and $y$-axes, respectively.

All abscissas measured to the *right* of the $y$-axis are taken as *positive,* while all abscissas measured to the *left* of the same axis are taken *negative.*

Thus the abscissa of $Q$ in Fig. 1 is $-2$; that of $R$ is $-3$; that of $S$ is $+3$.

Similarly, all ordinates *above* the $x$-axis are taken *positive,* while all ordinates *below* the same axis are taken *negative.*

Thus the ordinate of $Q$ is $+3$; that of $R$ is $-4$; that of $S$ is $-2$.

In reading the coordinates of a point, the abscissa is always read first and the ordinate second.

Thus, in Fig. 1, $P$ is the point $(3, 4)$; $Q$ is $(-2, 3)$; $R$ is $(-3, -4)$; $S$ is $(3, -2)$.

Let us now consider the following simple equation containing the *two* unknown numbers, $x$ and $y$:

$$(1) \qquad x+y=5.$$

Since any pair of values $(x, y)$ whose sum is 5 will satisfy this equation, it follows that there are an unlimited number of such $(x, y)$ pairs, or solutions. For example, the following four pairs are particular solutions:

$$(2) \quad (x=6, y=-1); (x=2, y=3); (x=\tfrac{1}{2}, y=\tfrac{9}{2}); (x=8, y=-3).$$

If we now regard each of these solutions as the coordinates of a point, and locate (plot) the four points thus obtained, it will be found that they all lie upon one and the same straight line, as shown in Fig. 2. This line is called the ***graph*** of the equation (1).

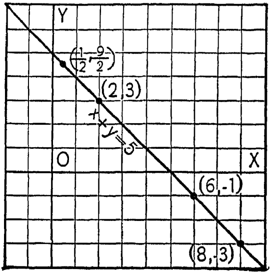

FIG. 2

Similarly, the graph of any simple (first degree) equation containing two letters may be drawn. However, it may be observed that in order to draw the graph it suffices to plot merely *two* solutions, since two points completely determine a line. Such a line will necessarily pass through, or contain, all the other solutions.

If, instead of one equation being given, there are *two* of them, as for example

$$(3) \qquad \begin{cases} x+y=6, \\ 3x-2y=-2, \end{cases}$$

and if we draw the graph of each, as in Fig. 3, then the point where the two graphs intersect will have as its coordinates a pair of values $(x, y)$ which satisfies *both* of the equations at once; in other words, it will give their common solution. In the present case this is seen to be the point $x=2$, $y=4$. This common solution is the same as would be obtained if one followed the method of elimination described in § 5. Hence, Fig. 3 may be regarded as giving the graphical meaning of such a solution.

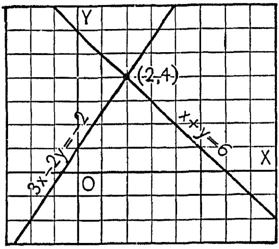

FIG. 3

**NOTE.** In exceptional cases, the graphs of two simple equations may turn out to be *parallel* lines so that they nowhere intersect. In such a case, the two equations have no common solution.

## EXERCISES

**1.** Plot (on coordinate paper) each of the following points.

(2, 4); (−2, 3); (−2, −4); ($2\frac{1}{2}$, −3); (0, −5); (4, 0); (0, 0).

**2.** Describe (*a*) the location of all points whose abscissa is zero; (*b*) of all points whose ordinate is zero; (*c*) of all points whose abscissa and ordinate are both negative.

Draw the graph of each of the following simple equations.

**3.** $x-y=5$. **5.** $2x-3y=1$. **7.** $x-3y=3$. **9.** $6x+7y=2$.

**4.** $2x+y=3$. **6.** $2x+3y=12$. **8.** $2x=3y$. **10.** $5x-8y=-1$.

Draw the graphs of each of the following pairs of equations and thus determine the values of $x$ and $y$ which form their common solution, if they have one. Check your results in each by solving by elimination (§ 5).

**11.** $\begin{cases} x+2y=3, \\ 2x+y=3. \end{cases}$ **14.** $\begin{cases} x-2y=3, \\ 2x-4y=1. \end{cases}$ **17.** $\begin{cases} x+2y=-1, \\ \frac{1}{2}x+y=2. \end{cases}$

**12.** $\begin{cases} x+y=3, \\ 3x-y=1. \end{cases}$ **15.** $\begin{cases} 4x-y=0, \\ 3x+y=7. \end{cases}$ **18.** $\begin{cases} 3x-3y=-5, \\ 3x+2y=40. \end{cases}$

**13.** $\begin{cases} x+2y=5, \\ x-2y=5. \end{cases}$ **16.** $\begin{cases} 4y-2x=5, \\ 4x+2y=5. \end{cases}$ **19.** $\begin{cases} 8x+4y=5, \\ x-y=\frac{1}{4}. \end{cases}$

**7. Literal Equations and Formulas.** Equations in which some, or all, of the known quantities are represented by letters are called ***literal equations.*** The known quantities are generally represented by the first letters of the alphabet, as $a$, $b$, $c$, etc. Literal equations are solved by the same processes as numerical equations.

EXAMPLE. Solve the following literal equation for $x$:

$$ax = bx + 7c.$$

SOLUTION. Transposing, we find

$$ax - bx = 7c.$$

Combining like terms, we have

$$(a-b)x = 7c.$$

Dividing by $(a-b)$, we obtain

$$x = \frac{7c}{a-b}. \quad \textit{Ans.}$$

It is to be noted that a literal equation is said to be solved for the unknown number, as $x$, only when this number has been expressed *in terms of* the other (known) letters, as illustrated in the preceding example.

An important special class of literal equations are the so-called *formulas* that occur in geometry, physics, engineering,

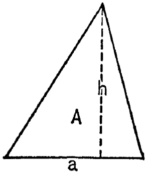

FIG. 4

etc. For example, if $a$ represents the length of the base of any triangle and $h$ represents the altitude, then the area, $A$, of the triangle is given (determined) by the formula

$$A = \frac{1}{2}ah. \tag{1}$$

Here the area is expressed *in terms of* the base and the altitude.

Similarly, the circumference, $C$, of any circle expressed in terms of the radius $r$ is given by the following formula, in which $\pi$ represents the incommensurable number, 3.1416 (approximately),

$$C = 2\pi r. \tag{2}$$

Again, the area, $A$, of a circle in terms of the radius $r$ is given by the formula

$$A = \pi r^2. \tag{3}$$

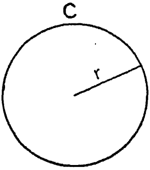

FIG. 5

As an example of an important formula in physics, it is readily seen that if an object moves during $t$ seconds with the constant velocity of $v$ feet per second, then the distance, $s$, passed over is given by the formula

$$s = vt. \tag{4}$$

Again, the following is an important formula in engineering: The horse-power, represented by $HP$, which a steam engine is delivering when the area of the piston is $A$ square inches, the pressure of the steam per square inch is $P$ pounds, the length of the piston stroke is $L$ feet and the number of strokes per minute is $N$, is given by the formula

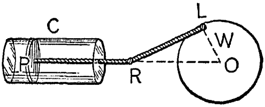

FIG. 6

$$HP = \frac{PLAN}{33,000}. \tag{5}$$

The following important formulas from plane and solid geometry are to be especially noted:

$$h^2 = a^2 + b^2, \tag{6}$$

which is the theorem of Pythagoras concerning the square of the hypotenuse of a right triangle.

$$V = \frac{4}{3}\pi r^3, \tag{7}$$

which gives the volume of a sphere in terms of its radius.

(8)
$$A=4\pi r^2,$$
which gives the area of a sphere in terms of its radius.

(9)
$$V=\pi r^2 h,$$
which gives the volume of a right circular cylinder in terms of the radius of the base, $r$, and the altitude, $h$.

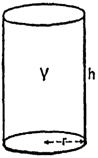

FIG. 7

(10)
$$A=2\pi rh,$$
which gives the area of the lateral surface of a right circular cylinder in terms of the radius of the base, $r$, and the altitude, $h$.

(11)
$$V=\frac{1}{3}\pi r^2 h,$$
which gives the volume of a cone of circular base, $r$, and of altitude, $h$.

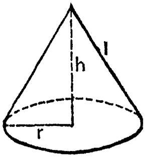

FIG. 8

(12)
$$A=\pi rl,$$
which gives the area of the lateral surface of a cone of circular base, $r$, and of slant-height, $l$.

(13)
$$V=\frac{\pi h}{2}\left[(a^2+b^2)+\frac{h^2}{3}\right],$$
which gives the volume of a spherical segment, or slice of a sphere between two parallel cutting planes, in terms of its altitude, $h$, and the radii, $a$, $b$, of its bases.

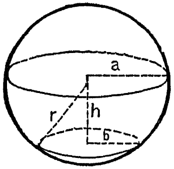

FIG. 9

(14)
$$V=2\pi rh,$$
which gives the area of a zone, or portion of the surface of a sphere lying between two parallel planes, in terms of the altitude, $h$, of the zone and the radius, $r$, of the sphere.

The following formulas from physics and mensuration may also be noted.

If an elastic ball (like a billiard ball) weighing $W_1$ ounces and moving with a velocity of $V_1$ feet per second strikes (impinges upon) a second ball of like size but weighing $W_2$

ounces, the latter standing at rest, then, after the impact, the first ball and the second ball will move with velocities $v_1$ and $v_2$ which are given respectively by the formulas

$$(15) \qquad v_1=\frac{W_1-W_2}{W_1+W_2}V_1 \text{ ft. per sec.,} \qquad v_2=\frac{2W_1}{W_1+W_2}V_1 \text{ ft. per sec.}$$

It is understood in the experiment just described that the first ball moves directly toward the center of the second one before the impact. Both continue in this same line after the impact.

### EXERCISES

Solve each of the following literal equations for $x$, checking your answer in the first five.

**1.** $ax-1=b.$

**2.** $ax+bx=c.$

**3.** $3x+b=x-3b.$

**4.** $\frac{x}{a}+b=\frac{x}{b}+a.$

**5.** $\frac{x-c}{c}+a=x-1.$

**6.** $\frac{x-b}{x-3}+\frac{x-c}{x+2}=2.$

**7.** $1-\frac{ab}{x}=\frac{7}{ab}-\frac{49}{abx}.$

**8.** $\frac{x-a}{b}+\frac{2x}{a}=5+\frac{6b}{a}.$

**9.** $\frac{a^2+b^2}{2bx}-\frac{a-b}{2bx^2}=\frac{b}{x}.$

**10.** $\frac{a-b+c}{x+a}=\frac{b-a+c}{x-a}.$

Solve (by the method of elimination) for $x$ and $y$ in each of the following pairs of equations.

**11.** $\begin{cases} 3x+5y=2a, \\ 2x-3y=4b. \end{cases}$

**12.** $\begin{cases} ax-by=2, \\ cx+dy=3. \end{cases}$

**13.** $\begin{cases} ax+by=m, \\ cx+dy=n. \end{cases}$

**14.** $\begin{cases} 3ax+2by=ab, \\ ax-by=ab. \end{cases}$

**15.** $\begin{cases} \frac{a}{x}-\frac{b}{y}=-1, \\ \frac{b}{x}-\frac{a}{y}=-1. \end{cases}$

[HINT. Solve first for $1/x$ and $1/y$. See Ex. 12, page 11]

**16.** Divide $a$ into two parts whose quotient is $m$.

**17.** If A can do a piece of work in $a$ days, and B can do it in $b$ days, how long will it take them if working together? (See Ex. 19, page 12.)

**18.** If the base of a triangle is 3 feet long, what must the altitude be in order that the area may be 30 square feet?

[HINT. Use formula (1).]

**19.** If the area of a circle is 44 square inches, what is the value (approximately) of the radius?

[HINT. Use formula (3), taking $\pi=3\frac{1}{7}$].

**20.** How long will it take a person to walk 1 mile if his rate of walking is 8 feet per second?

**21.** An automobile traveled $T$ hours at the rate of $v$ miles per hour. By how much would this rate have had to be increased in order that the distance be covered in $t$ minutes less time?

**22.** The formula for the area $A$ of a trapezoid whose bases are $B$ and $b$ and whose altitude is $h$ is

$$A=\tfrac{1}{2}h(B+b).$$

Using this formula, answer the following question: How long should the upper base of a trapezoid be in order that, if the lower base is 20 feet long and the altitude is 15 feet, the area may be 180 square feet?

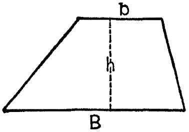

FIG. 10

**23.** The inside diameter of the cylinder of a steam engine is 8 inches, while the length of stroke is 1½ feet. When the steam gauge registers a pressure of 60 pounds per square inch, how many strokes per minute must the piston make if the engine is to deliver 22 horse-power? Work by formula (5).

**24.** The velocity, $v$, of sound, measured in feet per second, is given by the formula

$$v=1090+1.14(t-32),$$

where $t$ is the temperature of the air in Fahrenheit degrees.

Find ($a$) the velocity when the temperature is 75°; ($b$) the temperature when sound travels 1120 feet per second.

**25.** Derive formulas for the following:

($a$). The number $N$ of turns made by a wagon wheel $d$ feet in diameter in traveling $s$ miles.

($b$). The number $N$ of dimes in $m$ dollars, $n$ quarters and $q$ cents.

($c$). The value of a number containing three digits if the digit in unit's place is $a$, the digit in ten's place is $b$ and that in hundred's place is $c$.

**26.** The centrifugal force $F$, measured in pounds, with which a body weighing $W$ pounds pulls outward when moving in a circle of radius $r$ feet with a velocity of $v$ feet per second is determined by the formula

$$F = \frac{Wv^2}{32r}.$$

Use this formula to answer the following questions. A pail of water weighing 5 pounds is swung round at arm's length at a speed of 10 feet per second. If the length of the arm is 2 feet, find (*a*) the pull at the shoulder when the pail is at the uppermost point of its course; (*b*) when at the lowest point of its course. Also find the least velocity possible without water dropping out at the uppermost point of the course.

**27.** The weight $W$ that can be raised by the device shown in Fig. 11 is given by the formula

$$W = \frac{2\pi lRP}{dr}$$

where $P$ represents the pressure applied at the handle and where $R$, $r$, $d$ and $l$ have the lengths indicated in the figure. Show, by means of this formula, that if $d$ be halved and the number of teeth on the wheel be correspondingly doubled to fit the new gear, other parts remaining the same, then twice as much weight $W$ can be raised as before with a given pressure $P$ on the handle.

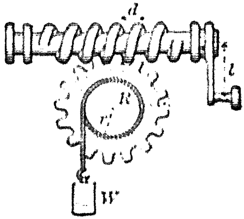

FIG. 11

## 8. Exponents.

The laws of exponents are briefly expressed in the following five formulas.

I. $$a^m a^n = a^{m+n}.$$

Thus $$4^3 \cdot 4^2 = 4^{3+2} = 4^5.$$

II. $$(a^m)^n = a^{mn}.$$

Thus $$(3^4)^5 = 3^{20}.$$

III. $$(ab)^m = a^m b^m.$$

Thus $$(2 \cdot 3)^2 = 2^2 \cdot 3^2.$$

**IV.** $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Thus $$\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}.$$

**V.** $$\frac{a^m}{a^n} = a^{m-n}.$$

Thus $$\frac{2^4}{2^2} = 2^{4-2} = 2^2.$$

These formulas apply not only when $m$ and $n$ are positive integers, but in all cases.

Thus $$2^{\frac{2}{3}} \cdot 2^{-\frac{1}{5}} = 2^{\frac{2}{3}-\frac{1}{5}} = 2^{\frac{7}{15}},$$

$$(3^{\frac{4}{5}})^{\frac{5}{6}} = 3^{\frac{4}{5} \times \frac{5}{6}} = 3^{\frac{2}{3}},$$

$$\frac{5^2}{5^4} = 5^{2-4} = 5^{-2}.$$

The use of formulas I – V in this universal way implies (as shown in elementary algebra) that the expression $a^{p/q}$ must have the same meaning as the *qth* root of $a^p$. That is, we have

**VI.** $$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Thus $$5^{\frac{2}{3}} = \sqrt[3]{5^2} = \sqrt[3]{25}.$$

Similarly, any quantity, as $a$, when raised to a *negative* exponent, as $-n$, must have the same meaning as the *reciprocal* of $a^n$. That is,

**VII.** $$a^{-n} = \frac{1}{a^n}.$$

Thus $$6^{-\frac{2}{3}} = \frac{1}{6^{\frac{2}{3}}}.$$

Again, for any value whatever of $a$, except 0, the expression $a^0$ has the value 1. That is, when $a$ is not 0

**VIII.** $$a^0 = 1.$$

## EXERCISES

Find the results of the indicated operations in each of the following cases, using one or more of the formulas I–V.

**1.** $2^5 \cdot 2^3$.
**2.** $(-1)^3(-1)^2$.
**3.** $\left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right)^4$.
**4.** $x^{10} \cdot x^2$.
**5.** $q^m\, q^4$.
**6.** $z^{r-1}.\ z^{r+1}$
**7.** $8^3 \div 8^2$.
**8.** $\left(\frac{4}{9}\right)^7 \div \left(\frac{4}{9}\right)^5$.
**9.** $x^{10} \div x^2$.
**10.** $q^m \div q^4$.
**11.** $z^{r+1} \div z^{r-1}$.
**12.** $(2^2)^3$.
**13.** $\left\{(-2)^3\right\}^2$.
**14.** $(x^6)^4$.
**15.** $(a^2b^3)^3$.
**16.** $(x^2y^2)^2$.
**17.** $(m^2n^3w)^3$.
**18.** $\left\{(a+b)^2(c+d)^3\right\}^4$.
**19.** $\left(\frac{m^5}{n}\right)^4$.
**20.** $\left(\frac{x^2}{y}\right)^3 \cdot \left(\frac{x}{y^2}\right)^3$.
**21.** $\left(-\frac{x^{2n}}{y^{3m}}\right)^k$.
**22.** $3^{-2}$.
**23.** $2^{-1} \cdot 3^{-2}$.
**24.** $4^0 \cdot 3^{-3}$.
**25.** $(-8)^{-\frac{1}{3}}$.
**26.** $a^{-2} \cdot a^{-5}$.
**27.** $n^2 \cdot bn^{-3}$.
**28.** $(a^{\frac{1}{2}}+b^{\frac{1}{2}})a^{\frac{1}{2}}b^{\frac{1}{2}}$.
**29.** $x^{\frac{3}{2}} \div x^{-\frac{1}{2}}$.
**30.** $(x^{-\frac{1}{3}})^6$.
**31.** $\left(\frac{1}{8}x^9\right)^{\frac{1}{3}}$.

Write each of the following expressions with a radical sign, and then simplify as far as possible.

**32.** $8^{\frac{2}{3}}$.

SOLUTION. $8^{\frac{2}{3}} = \sqrt[3]{8^2}$ (Formula VI)

$= \sqrt[3]{64} = 4.$ *Ans.*

**33.** $8^{\frac{1}{3}}$.
**34.** $9^{\frac{1}{2}}$.
**35.** $(-8)^{\frac{1}{3}}$.
**36.** $27^{\frac{2}{3}}$.
**37.** $81^{\frac{3}{4}}$.
**38.** $64^{\frac{1}{6}}$.
**39.** $(x^6)^{\frac{1}{3}}$.
**40.** $(y^{10})^{\frac{2}{5}}$.
**41.** $2^{\frac{2}{3}}$.
**42.** $m^{\frac{2}{3}}n^{\frac{3}{4}}$.

**43.** Solve the equation $x^{-\frac{3}{2}} = 27$.

SOLUTION. Raising both members to the power $-\frac{2}{3}$, we have

$$(x^{-\frac{3}{2}})^{-\frac{2}{3}} = 27^{-\frac{2}{3}},$$

or (using formula II)

$$x^1 = 27^{-\frac{2}{3}},$$

or (using formula VII)

$$x = \frac{1}{27^{\frac{2}{3}}}.$$

This answer for $x$ may be simplified by noticing that we may write $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$. Hence the final answer is $\frac{1}{9}$.

**44.** Solve for $x$ in each of the following equations.

(a) $x^{\frac{1}{2}}=2$.
(c) $x^{-\frac{3}{5}}=-\frac{1}{8}$.
(e) $x^{\frac{5}{3}}+32=0$.
(b) $x^{\frac{3}{2}}=27$.
(d) $x^{-\frac{1}{n}}=-3$.
(f) $\frac{1}{4}x^{\frac{3}{4}}=\frac{1}{25}$.

**45.** Multiply $x+3x^{\frac{2}{3}}-2x^{\frac{1}{3}}$ by $3-2x^{-\frac{1}{3}}+4x^{-\frac{2}{3}}$.

SOLUTION.

$$\begin{array}{rrrrr}
x+3x^{\frac{2}{3}}-2x^{\frac{1}{3}} & & & & \\
3-2x^{-\frac{1}{3}}+4x^{-\frac{2}{3}} & & & & \\
3x+9x^{\frac{2}{3}} & -6x^{\frac{1}{3}} & & & \\
-2x^{\frac{2}{3}} & -6x^{\frac{1}{3}} & +4 & & \\
 & +4x^{\frac{1}{3}} & +12 & -8x^{-\frac{1}{3}} & \\
3x+7x^{\frac{2}{3}} & -8x^{\frac{1}{3}} & +16 & -8x^{-\frac{1}{3}}. & \textit{Product.}
\end{array}$$

Multiply

**46.** $a-2a^{\frac{1}{2}}+3$ by $2a^{\frac{1}{2}}+3$.

**47.** $2x^{\frac{2}{3}}-3x^{\frac{1}{3}}-4+x^{-\frac{1}{3}}$ by $3x^{\frac{1}{3}}+x-2x^{-\frac{1}{3}}$.

**48.** $a^{\frac{2}{3}}x^{-\frac{3}{4}}+2+a^{-\frac{2}{3}}x^{\frac{3}{4}}$ by $2a^{-\frac{2}{3}}x^{\frac{3}{4}}-4a^{-\frac{4}{3}}x^{\frac{3}{2}}+2a^{-2}x^{\frac{9}{4}}$.

Divide

**49.** $5x+2x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1$ by $x^{\frac{1}{3}}+1$.

**50.** $x^{-\frac{1}{2}}-x^{-\frac{1}{6}}+5-2x^{\frac{1}{6}}$ by $1+2\sqrt[6]{x}$.

**51.** $\frac{x}{y}-x^{\frac{2}{3}}y^{-\frac{1}{2}}-4\sqrt[3]{x}-\frac{8y}{\sqrt[3]{x}}$ by $\sqrt[3]{x}+2y^{\frac{1}{2}}$.

**9. Simplification of Radicals.** We know that the squa root of the product of two numbers is the same as the produ of their square roots. For example, $\sqrt{4\cdot 25}$ is the same $\sqrt{4}\times\sqrt{25}$, because both are equal to 10, for the first is $\sqrt{10}$ or 10, and the second is $2\times 5$, or 10. In the same wa $\sqrt[3]{8\times 3}=\sqrt[3]{8}\times\sqrt[3]{3}$, or simply $2\sqrt[3]{3}$. In fact, we have t following general formula

IX. $$\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}.$$

Again, $\sqrt{4/9}$ is the same as $\sqrt{4}/\sqrt{9}$, because both equal to 2/3. (Explain.) Similarly, $\sqrt{5/8}$ may be writt

$\sqrt[3]{5}/\sqrt[3]{8}$, which reduces to the more simple form $\frac{1}{2}\sqrt[3]{5}$. So in general we have

$$\text{X.}\qquad \sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Note that formula IX may be regarded as a special consequence of formula III, while formula X is a special consequence of formula IV.

The two preceding formulas enable us to simplify many radical expressions, as illustrated below.

EXAMPLE 1. Simplify $\sqrt{54}$.

SOLUTION. Using formula IX, we have

$$\sqrt{54}=\sqrt{9\times 6}=\sqrt{9}\times\sqrt{6}=3\sqrt{6}.\quad \textit{Ans.}$$

EXAMPLE 2. Simplify $\sqrt[3]{32}$.

SOLUTION. $\sqrt[3]{32}=\sqrt[3]{8\times 4}=\sqrt[3]{8}\times\sqrt[3]{4}=2\sqrt[3]{4}$. *Ans.*

EXAMPLE 3. Simplify $\sqrt{\frac{8}{27}}$.

SOLUTION. $\sqrt{\frac{8}{27}}=\frac{\sqrt{8}}{\sqrt{27}}=\frac{\sqrt{4}\times\sqrt{2}}{\sqrt{9}\times\sqrt{3}}=\frac{2\sqrt{2}}{3\sqrt{3}}$. *Ans.*

EXAMPLE 4. Simplify $\sqrt{28a^6b}$.

SOLUTION. $\sqrt{28a^6b}=\sqrt{4a^6\times 7b}=\sqrt{4a^6}\times\sqrt{7b}=2a^3\sqrt{7b}$. *Ans.*

EXAMPLE 5. Simplify $\sqrt[3]{\frac{72x^2y^6}{z^6}}$.

SOLUTION. $\sqrt[3]{\frac{72x^2y^6}{z^6}}=\frac{\sqrt[3]{8y^6\times 9x^2}}{\sqrt[3]{z^6}}=\frac{\sqrt[3]{8y^6}\times\sqrt[3]{9x^2}}{z^2}=\frac{2y^2\sqrt[3]{9x^2}}{z^2}$. *Ans.*

## EXERCISES

Simplify each of the following radicals.

**1.** $\sqrt{18}$.
**2.** $\sqrt{24}$.
**3.** $\sqrt{72}$.
**4.** $\sqrt{125}$.
**5.** $\sqrt{99}$.
**6.** $\sqrt[3]{32}$.
**7.** $\sqrt[3]{54}$.
**8.** $\sqrt[3]{81}$.
**9.** $\sqrt[4]{32}$.
**10.** $\sqrt{\frac{72}{75}}$.
**11.** $\sqrt[3]{\frac{24}{135}}$.
**12.** $\sqrt{36a^5b^3}$.
**13.** $\sqrt{81m^6n^7}$.
**14.** $\sqrt{4(a+b)^3}$.
**15.** $\sqrt[3]{27x^4y^3z^2}$.
**16.** $\sqrt{\frac{16h^2k^4}{s^3t}}$.
**17.** $\sqrt[3]{\frac{16h^2k^4}{s^3t}}$.
**18.** $\sqrt{\frac{3(a+b)^2c^2d}{4(a^2-b^2)}}$.

**19.** Reduce

$$\frac{1-\sqrt{2}}{\sqrt{6}}$$

to an equivalent fraction having no radical in its denominator.

SOLUTION. Multiply both numerator and denominator by $\sqrt{6}$, thus obtaining

$$\frac{\sqrt{6}-\sqrt{12}}{6}, \text{ or } \frac{\sqrt{6}-2\sqrt{3}}{6}. \quad \text{Ans.}$$

Reduce each of the following expressions to equivalent fractions having no radicals in their denominators.

**20.** $\dfrac{2+\sqrt{5}}{2\sqrt{7}}$.

**21.** $\dfrac{3\sqrt{2}-\sqrt{3}}{2\sqrt{6}}$.

**22.** $\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$.

[HINT TO EX. 22. Multiply both numerator and denominator by $3-\sqrt{2}$.]

**23.** $\dfrac{3\sqrt{a}-4\sqrt{b}}{2\sqrt{a}-3\sqrt{b}}$.

**24.** $\dfrac{\sqrt{x+1}+3}{\sqrt{x+1}+2}$.

**25.** $\dfrac{2\sqrt{2a-1}+3\sqrt{a}}{3\sqrt{2a-1}+2\sqrt{a}}$.

**26.** $\dfrac{\sqrt{a}+\sqrt{b}-\sqrt{a+b}}{\sqrt{a}-\sqrt{b}+\sqrt{a+b}}$.

**10. Imaginary Numbers. Complex Numbers.** An indicated square root of a negative quantity, as for example $\sqrt{-4}$ or $\sqrt{-1/2}$, is called an ***imaginary number***, or a ***pure imaginary number***. Such a combination as $5+\sqrt{-4}$, wherein a pure imaginary is added to an ordinary (real) number, is called a ***complex number***. Every complex number can be reduced to the typical form $a+b\sqrt{-1}$, where $a$ and $b$ are properly determined real (positive or negative) numbers. Thus $5+\sqrt{-4}$ becomes $5+2\sqrt{-1}$; likewise $7-\sqrt{-3}$ becomes $7-\sqrt{3}\sqrt{-1}$; etc. In all problems involving complex numbers, first put each in its proper form $a+b\sqrt{-1}$, then proceed according to the customary rules of algebra, remembering to substitute for $(\sqrt{-1})^2$, wherever it occurs, the value $-1$.

In the exercises which follow, the symbol $i$ is used for brevity to stand for $\sqrt{-1}$.

## EXERCISES

Perform the indicated operations and simplify where possible.

**1.** $(1+i)(2-i)$.

Solution. $(1+i)(2-i)=2+i-i^2=2+i-(-1)=3+i$. *Ans.*

**2.** $(1+\sqrt{-2})(2-\sqrt{-3})$.

[Hint. First write as $(1+\sqrt{2}\,i)(2-\sqrt{3}\,i)$. Now proceed as in Ex. 1, obtaining the final result $(2+\sqrt{6})+i\,(2\sqrt{2}-\sqrt{3})$.]

**3.** $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$. **4.** $(1+\sqrt{-2})^2-(1-\sqrt{-2})^2$.

Express each of the following fractions with a denominator containing no radicals:

**5.** $\dfrac{1}{3-\sqrt{-2}}$. **6.** $\dfrac{2-\sqrt{-3}}{2+\sqrt{-3}}$. **7.** $\dfrac{a+bi}{a-bi}$.

**8.** If $x=\dfrac{-1+\sqrt{-2}}{3}$, show that $3x^2+2x+1=0$. It follows that the indicated value of $x$ is a *root* of the given equation. Such roots are called *imaginary roots*.

**9.** Show that in each of the following the value given for $x$ is an imaginary root (see Ex. 8) of the corresponding equation.

(*a*) $x=\dfrac{-1+\sqrt{-3}}{2}$; $x^2+x+1=0$.

(*b*) $x=\dfrac{3+\sqrt{-7}}{4}$; $2x^2-3x+2=0$.

(*c*) $x=\dfrac{-5-\sqrt{-3}}{4}$; $4x^2+10x+7=0$.

**10.** Is $x=2+\sqrt{-3}$ a root of $x^2+2x+3=0$? Why?

*For a further study of complex numbers see Chapter* XV, *page* 231.

# CHAPTER II

## QUADRATIC EQUATIONS†

**11. Solution by Inspection.** PROBLEM. It is desired to cut out a rectangle which shall contain 4 square inches and be 3 inches longer than wide. What must be its dimensions (length and breadth)?

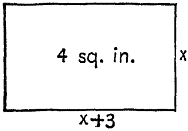

FIG. 12

SOLUTION. Let $x$ represent the breadth. Then $x+3$ will be the length and, by the rule for determining the area of a rectangle, we shall have

$$x(x+3)=4, \tag{1}$$

or

$$x^2+3x=4. \tag{2}$$

We here meet with what is known as a ***quadratic equation,*** that is, one containing the square (but no higher power) of the unknown quantity $x$. Moreover, we see by inspection that the value $x=1$ satisfies this equation, since with $x=1$ the left side becomes $1^2+3\cdot 1$, which reduces to 4 as required. The dimensions sought are, therefore, 1 inch and 1 inch + 3 inches, or 4 inches. *Ans.*

**12. Completing the Square in a Quadratic Equation.** Suppose now that in the problem of § 11, we require that the area shall be 5 square inches instead of 4 square inches, other conditions remaining the same. Then the equation which we shall have to solve will evidently be [compare (2)]

$$x^2+3x=5. \tag{3}$$

This equation is not easily solved by inspection, as was done with (2), but it can be solved, as we shall now show, by an ingenious method known as ***completing the square.***

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† This chapter may be omitted by those already familiar with the elements of quadratic equations. Such students may pass at once to Chapter III, which deals with the general properties of such equations.

Add 9/4 to each member of (3), giving

$$(4)\qquad x^2+3x+\frac{9}{4}=5+\frac{9}{4},\ \text{ or }\ x^2+3x+\frac{9}{4}=\frac{29}{4}.$$

Here the left member is the same as $\left(x+\frac{3}{2}\right)^2$, since by the familiar formula

$$(a+b)^2=a^2+2ab+b^2$$

we obtain

$$\left(x+\frac{3}{2}\right)^2=x^2+3x+\frac{9}{4}.$$

Thus, (4) may be written

$$(5)\qquad \left(x+\frac{3}{2}\right)^2=\frac{29}{4}.$$

Equation (3) has now taken a form (5) wherein the left member is a perfect square. Consequently we have only to extract the square root of each member of (5) in order to obtain

$$(6)\qquad x+\frac{3}{2}=\sqrt{\frac{29}{4}},\qquad \text{or}\qquad x+\frac{3}{2}=\frac{1}{2}\sqrt{29}.$$

from which it follows that

$$(7)\qquad x=\frac{1}{2}\sqrt{29}-\frac{3}{2}\qquad \text{or}\qquad x=\frac{1}{2}(\sqrt{29}-3).$$

Substituting for $\sqrt{29}$ its approximate value 5.385, as given in the Tables, we have finally

$$x=\frac{1}{2}(5.385-3)=\frac{1}{2}\text{ of }2.385=1.192\text{ (approximately).}$$

Hence the required dimensions of the rectangle are (approximately) 1.192 inch and 1.192 inch + 3 inches = 4.192 inches. *Ans.*

These values for the two dimensions are correct to three places of decimals. The *exact* values, of course, cannot be expressed decimally, since $\sqrt{29}$ cannot be so expressed.

**13. The Two Solutions of a Quadratic Equation.** It is important to observe at this point that if we inquire simply what values of $x$ satisfy the quadratic equation (3), that is, without any reference to the rectangle, we may find *two* such values. In fact, in passing from equation (5) to (6), we should remember that the square root of 29/4 is either $+\frac{1}{2}\sqrt{29}$ or $-\frac{1}{2}\sqrt{29}$, since either of these when squared gives 29/4. If we take the value $+\frac{1}{2}\sqrt{29}$ we get (6), which leads to the value of $x$ given in (7), but if we take $-\frac{1}{2}\sqrt{29}$, we get instead of (6) the equation

$$x+\frac{3}{2}=-\frac{1}{2}\sqrt{29}, \tag{8}$$

from which we obtain

$$x=-\frac{1}{2}\sqrt{29}-\frac{3}{2}=-\frac{1}{2}(\sqrt{29}+3)=-4.192 \text{ (approx.)}. \tag{9}$$

In reality, then, there are two values of $x$ which satisfy (3), namely $\frac{1}{2}(\sqrt{29}-3)$ and $-\frac{1}{2}(\sqrt{29}+3)$. These taken together are called the ***roots*** of the equation. In the rectangle problem of § 12 we could not use the second of these roots since it is a negative quantity, and there would be no meaning to a rectangle having negative dimensions. However, problems frequently arise in which we can use both roots, as will be illustrated presently.

For convenience, the symbol ±, read plus or minus, is frequently used in expressing the two roots of a quadratic equation. Thus, the two roots of (1) may now be expressed concisely in the form

$$x=-\frac{3}{2}\pm\frac{1}{2}\sqrt{29}, \quad \text{or} \quad x=\frac{1}{2}(-3\pm\sqrt{29}).$$

**14. Application to Any Quadratic Equation.** A careful examination of the process followed in §§ 12, 13 for arriving at the two roots of the special quadratic equation

$$x^2+3x=5$$

shows that what was added to both sides in order to complete the square on the left was 9/4, which is $(3/2)^2$, or the square of half the coefficient of the first power of $x$ in the given equation. More generally, it is now to be observed that if we have *any* quadratic equation of the form

$$x^2+mx=n, \tag{10}$$

where the coefficients $m$ and $n$ are given numbers, we may likewise complete the square and solve by adding to both members the square of half the coefficient of the first power of $x$; that is, by adding $(m/2)^2$. In fact, we thus obtain from (10) the equation

$$x^2+mx+\left(\frac{m}{2}\right)^2=n+\left(\frac{m}{2}\right)^2,$$

or

$$\left(x+\frac{m}{2}\right)^2=n+\left(\frac{m}{2}\right)^2,$$

after which we may evidently proceed in all cases to solve as in steps (5), (6), (7) and (8) of §§ 12 and 13. Thus we may state the following rule.

Rule. *In order to solve any quadratic equation of the form*

$$x^2+mx=n$$

*first complete the square of the left member by adding the square of half the coefficient of $x$ to both sides of the equation. Take the square root of both members of the resulting equation, giving the sign ± to the right member of the result. Solve the two first degree equations thus obtained for $x$.*

It is to be observed that this rule applies only in case the coefficient of $x^2$ in the given equation is 1. It does not apply, for example, to the equation

$$3x^2+5x=12.$$

However, in this case we have only to divide the equation through by the coefficient of $x^2$, namely 3, in order to cause the equation to take the form

$$x^2+\frac{5}{3}x=4,$$

and to this the rule may now be applied directly, inasmuch as the coefficient of $x^2$ is 1. Similarly, any quadratic equation whatever may either be solved directly by the rule or after division of both members by the coefficient of $x^2$ in case this coefficient is different from 1. Illustrative examples of both these species of applications occur below.

Example 1. Solve the quadratic equation

$$x^2-10x=5.$$

Solution. Here, the coefficient of $x^2$ being 1, we may make direct application of the rule. Thus, the coefficient of $x$ is $-10$ so that the square of half this number, which is $(-5)^2$ or 25, is to be added to both members, giving the equation

$$x^2-10x+25=30$$

which may be written

$$(x-5)^2=30.$$

Hence, extracting the square root of both members, we have

$$x-5=\pm\sqrt{30}.$$

The two desired roots are therefore

$$x=5\pm\sqrt{30}.$$

Check. Placing the root $5+\sqrt{30}$ for $x$ in the left member of the given equation, we obtain

$$(5+\sqrt{30})^2-10(5+\sqrt{30})=25+10\sqrt{30}+30-50-10\sqrt{30}$$

which, upon noting cancellation, reduces to the right member, or 5.

Similarly, for the other root, $5-\sqrt{30}$, we have

$$(5-\sqrt{30})^2-10(5-\sqrt{30})=25-10\sqrt{30}+30-50+10\sqrt{30}=5.$$

EXAMPLE 2. Solve the quadratic equation

$$2x^2-3x-9=0.$$

SOLUTION. The coefficient of $x^2$ being 2, we first divide through by 2, transposing also the $-9$, in order to obtain an equation of the species mentioned in the rule. Thus, our equation becomes

$$x^2-\tfrac{3}{2}x=\tfrac{9}{2}.$$

The rule may now be applied directly, the details being as follows: Completing the square by adding $(-3/4)^2$, or $9/16$, to both sides,

$$x^2-\frac{3}{2}x+\frac{9}{16}=\frac{9}{2}+\frac{9}{16}=\frac{81}{16}.$$

Extracting the square root of each side,

$$x-\frac{3}{4}=\pm\sqrt{\frac{81}{16}}=\pm\frac{9}{4}.$$

Hence the roots are

$$x=\frac{3}{4}\pm\frac{9}{4}\text{, that is } x=3 \text{ and } x=-\frac{3}{2}.$$

CHECK. $2\cdot 3^2-3\cdot 3=2\cdot 9-9=18-9=9.$

Again, $2\left(-\dfrac{3}{2}\right)^2-3\left(-\dfrac{3}{2}\right)=2\cdot\dfrac{9}{4}-3\left(-\dfrac{3}{2}\right)=\dfrac{9}{2}+\dfrac{9}{2}=9.$

### ORAL EXERCISES

In each of the following equations, state what must be added to each side of the equation in order to complete the square.

**1.** $x^2+4x=5.$
**2.** $x^2+5x=-3.$
**3.** $x^2-7x=1.$
**4.** $x^2+18x=1\tfrac{1}{2}.$
**5.** $x^2-1\tfrac{2}{3}x=5.$
**6.** $x^2+\tfrac{2}{3}x=\tfrac{7}{9}.$
**7.** $x^2+2ax=10.$
**8.** $x^2+2(a+b)x=c.$
**9.** $x^2+(a+b)x=c.$
**10.** $x^2-(m-n)x=p.$

### EXERCISES

Find the two roots of each of the following equations and check your answers. Whenever radicals present themselves, evaluate each root correct to three decimal places by use of the Tables.

**1.** $x^2+2x=1.$
**2.** $x^2+6x=16.$
**3.** $x^2-8x-20=0.$
**4.** $3x^2+4x=7.$
**5.** $5x^2-6x=8.$
**6.** $3x^2+7x=26.$
**7.** $2x+3\tfrac{3}{4}x^2=4.$
**8.** $x^2+5=\dfrac{14x}{3}.$
**9.** $3x^2-5x=-1.$

**10.** $9x^2-18x+4=0.$

**11.** $3x^2+\frac{1}{2}x=1\frac{2}{3}.$

**12.** $\frac{9}{5x}-1+\frac{2}{x+2}=0.$

[Hint. First clear of fractions.]

**13.** $\frac{3x+5}{x+4}=3-\frac{2x-5}{x-2}.$

**14.** $\frac{2}{3x-1}+\frac{3x}{2x-5}=0.$

**15.** $\frac{2x-1}{x+1}-\frac{3x-4}{x-1}=1-\frac{4x-14}{1-x^2}.$

**16.** $2x^2+5x=-4.$

**17.** $3x^2-7x=-5.$

**18.** $3x(x+1)-(x-2)(x+3)=2+(1-x)^2.$

**19.** $\frac{x^2+x-1}{x^2+x+1}+\frac{x^2-x-1}{x^2-x+1}=2.$

**20.** $3(2x-5)(x+1)-2(3x+2)(2x-3)=x-9.$

**15. Literal Quadratic Equations.** Such equations are solved by the same methods as employed in solving quadratic equations with numerical coefficients.

Example 1. Solve the equation

$$2x^2-ax=\frac{a}{2}(x+a).$$

Solution. Clearing of fractions

$$4x^2-2ax=ax+a^2.$$

Hence

$$4x^2-3ax=a^2,$$

or

$$x^2-\frac{3a}{4}x=\frac{a^2}{4}.$$

Completing the square, following the rule in § 14,

$$x^2-\frac{3a}{4}x+\left(\frac{3a}{8}\right)^2=\frac{a^2}{4}+\left(\frac{3a}{8}\right)^2=\frac{a^2}{4}+\frac{9a^2}{64}=\frac{25a^2}{64}.$$

Extracting the square root of each side,

$$x-\frac{3a}{8}=\pm\frac{5a}{8}.$$

Whence the two roots are

$$x=\frac{3a}{8}\pm\frac{5a}{8}.$$

That is, the two roots are

$$x=\frac{3a+5a}{8}=a \text{ and } x=\frac{3a-5a}{8}=-\frac{a}{4}.$$

Example 2. Solve the equation

$$\frac{x}{x-1}-\frac{x}{x+1}=m.$$

Solution. Clearing of fractions, we have

$$x(x+1)-x(x-1)=m(x-1)(x+1),$$

or

$$x^2+x-x^2+x=mx^2-m.$$

Hence

$$mx^2-2x=m,$$

or

$$x^2-\frac{2}{m}x=1.$$

Completing the square,

$$x^2-\frac{2}{m}x+\left(\frac{1}{m}\right)^2=1+\left(\frac{1}{m}\right)^2=\frac{m^2+1}{m^2}.$$

Extracting the square root of each side,

$$x-\frac{1}{m}=\pm\frac{1}{m}\sqrt{m^2+1}.$$

Hence, the two roots are

$$x=\frac{1}{m}\pm\frac{1}{m}\sqrt{m^2+1}\,;\ \text{or}\ \frac{1}{m}(1\pm\sqrt{m^2+1}).$$

Since $m^2+1$ is not a perfect square, these roots cannot be further simplified.

## EXERCISES

Solve each of the following equations for $x$, reducing your answer to its simplest form.

**1.** $x^2+4ax=12a^2$.

**2.** $x^2+4bx=21b^2$.

**3.** $5ax+6a^2=6x^2$.

**4.** $21b^2-4bx=x^2$.

**5.** $3x^2+4cdx=15c^2d^2$.

**6.** $x^2+\frac{5x}{a}=\frac{6}{a^2}$.

**7.** $x^2+2mx=m^2$.

**8.** $x^2+2mx=m$.

**9.** $x^2-(a+1)x+a=0$.

**10.** $ax^2-(a^2-1)x=a$.

**11.** $\frac{x+1}{x^2}=\frac{a+1}{a^2}$.

**12.** $x^2+\frac{a}{b}x=\frac{a+b}{b}$.

**13.** $x-\frac{1}{x-a}=a.$

**14.** $4(x^2-1)=b(4x-b).$

**15.** $a(2x-1)+2bx-b=x(2x-1).$

**16.** $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}.$

**17.** $abx^2=\frac{1}{ab}\left[x(a+b)-\frac{1}{ab}\right].$

**18.** $\frac{x^2+1}{x}=\frac{a+b}{c}+\frac{c}{a+b}.$

**19.** $\frac{2a+x}{2a-x}+\frac{a-2x}{a+2x}=\frac{8}{3}.$

**20.** $\frac{x^2}{a+b}-\left(1+\frac{1}{ab}\right)x+\frac{1}{a}+\frac{1}{b}=0.$

**16. Solution by Factoring.** If, after arranging a quadratic equation so that its right member is zero, it is found that the left member can be readily factored, the roots of the equation can be obtained immediately without completing the square. The principle employed is the familiar one of arithmetic that *if any one factor of a product equals zero, then the product itself equals zero.*

EXAMPLE. Solve the equation

$$x^2+x=6.$$

SOLUTION. Rewriting so that the right member is zero,

$$x^2+x-6=0.$$

Factoring,

$$(x-2)(x+3)=0.$$

This equation will be satisfied, according to the above principle, in case either $x-2=0$ or $x+3=0$; that is, in case $x=2$ or $x=-3$. The roots desired are therefore 2 and $-3$.

**17. Solving Equations of Higher Degree.** Since the principle stated in § 16 applies to the product of any number of factors, equations of higher degree than the second frequently may be solved by this method.

EXAMPLE 1. Solve the equation

$$x(x-1)(x+2)(x-4)=0.$$

SOLUTION. Since the right member is 0, the equation will be satisfied in case $x=0$, or $x-1=0$ or $x+2=0$ or $x-4=0$. Hence the roots are 0, 1, $-2$ and 4.

EXAMPLE 2. Solve the equation $x^3-1=0$. Factoring, we find

$$(x-1)(x^2+x+1)=0.$$

The equation $x-1=0$ gives $x=1$ as one root.

The equation $x^2+x+1=0$ is a quadratic equation whose roots are found (by completing the square) to be $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

The roots desired are therefore 1 and $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

EXAMPLE 3. Solve the equation

$$x^3+x^2=4(1+x).$$

We have

$$x^3+x^2-4(1+x)=0,$$

or

$$x^2(x+1)-4(x+1)=0.$$

Factoring,

$$(x+1)(x^2-4)=0, \text{ or } (x+1)(x-2)(x+2)=0.$$

Hence the roots are $-1$, 2 and $-2$.

## EXERCISES

Solve each of the following equations by factoring.

**1.** $x^2+5x+6=0$.

**2.** $x^2-6x=27$.

**3.** $6x^2-x-15=0$.

[HINT. Factor into $(3x-5)(2x+3)=0$.]

**4.** $4x^2+5x-6=0$.

**5.** $12x^2-5x=3$.

**6.** $3a^2x^2+10ax=8$.

**7.** $x^4=16$.

[HINT TO EX. 7. Write first in form $(x^2-4)(x^2+4)=0$.]

**8.** $x^3=27$.

[HINT. Recall the formula $x^3-a^3=(x-a)(x^2+ax+a^2)$.]

**9.** $x^4-5x^2+4=0$.

**10.** $(2x-1)(5x^2+4x-3)=0$.

**11.** $3(x^2-1)-2(x+1)=0$.

**12.** $8(x^3-1)+3(x-1)=0$.

**13.** $2x^3+2x^2=x+1$.

**14.** $x^2-(a+b)x+ab=0$.

**18. Equations in Quadratic Form.** If an equation may be brought into the form of a quadratic by the use of a new letter it is said to be an equation in *quadratic form.* Having once brought it into such a form, its solution is readily effected by the methods already explained.

Example 1. Solve the equation

$$x^4-13x^2+36=0.$$

Solution. Let $x^2=y$. Substituting $y$ for $x^2$ in the equation, we obtain

$$y^2-13y+36=0.$$

Solving, we find that the roots of this quadratic are

$$y=4 \text{ and } y=9.$$

Hence $x^2=4$ and $x^2=9$. Therefore $x=\pm 2$ and $x=\pm 3$, and these are the desired roots of the original equation.

Example 2. Solve the equation

$$(2x-3)^2-6(2x-3)=7.$$

Solution. Substituting $y$ for $2x-3$, we obtain

$$y^2-6y=7.$$

Solving,

$$y=7 \text{ and } y=-1.$$

Hence

$$2x-3=7 \text{ and } 2x-3=-1.$$

Therefore

$$x=5 \text{ and } x=1. \quad \textit{Ans.}$$

Example 3. Solve the equation $x+\sqrt{x}=12$.

Solution. Substituting $y$ for $\sqrt{x}$, we obtain

$$y^2+y=12.$$

Whence, solving,

$$y=-4 \text{ and } y=3.$$

Therefore

$$\sqrt{x}=-4 \text{ and } \sqrt{x}=3.$$

Of these two possible values of $\sqrt{x}$, we are here obliged to throw out the value $-4$, because the form of our original equation implies that, whatever $x$ may be, its positive square root is to be used. Otherwise, the equation would have read $x-\sqrt{x}=12$.

From the remaining possibility, namely $\sqrt{x}=3$, we obtain upon squaring, $x=9$.

Therefore the equation has one root, namely $x=9$.

CHECK. $$9+\sqrt{9}=9+3=12.$$

Note that if an equation contains radicals, care must be taken, as illustrated in the above solution of Ex. 3, to retain only those roots which satisfy the equation when each of its radicals is taken with its indicated sign. This will be further illustrated in § 19.

### EXERCISES

Solve, by the method of substitution employed in § 18, each of the following equations and verify your answer in each case.

**1.** $x^4-5x^2+4=0.$

**2.** $x^4-7x^2+12=0.$

**3.** $4x^4-13x^2+9=0.$

**4.** $27x^6-35x^3+8=0.$

**5.** $(x-2)^2+2(x-2)=3.$

**6.** $(x^2-2)^2+2(x^2-2)=8.$

**7.** $3x^{\frac{1}{2}}-5x^{\frac{1}{4}}=2.$

**8.** $x-5+2\sqrt{x-5}=8.$

**9.** $x-3=21-4\sqrt{x-3}.$

**10.** $\dfrac{x^2}{x+1}+\dfrac{x+1}{x^2}=\dfrac{25}{12}.$

[HINT. Let $\dfrac{x^2}{x+1}=y.$]

**11.** $\dfrac{x}{x^2+1}+\dfrac{x^2+1}{x}=\dfrac{5}{2}.$

**12.** $2x-6\sqrt{2x-1}=8.$

[HINT. Adding $-1$ to both members, we obtain

$$(2x-1)-6\sqrt{2x-1}=7.]$$

**13.** $x=11-3\sqrt{x+7}.$

**14.** $3x^{-\frac{2}{3}}+5x^{-\frac{1}{3}}=2.$

**15.** $3x^{-\frac{1}{2}}-7x^{\frac{1}{4}}=4.$

**16.** $2x^2-\sqrt{x^2-2x-3}=4x+9.$

**17.** $\sqrt{\dfrac{7-2x}{7+2x}}+\sqrt{\dfrac{7+2x}{7-2x}}=\dfrac{3}{2}\sqrt{2}.$

**19. Radical Equations.** Equations containing radicals are frequently called *radical equations*. They may often be solved in the manner illustrated below.

EXAMPLE. Solve the equation

$$\sqrt{2x+5}-\sqrt{x+2}=\sqrt{x-1}.$$

SOLUTION. Squaring

$$2x+5-2\sqrt{(2x+5)(x+2)}+x+2=x-1.$$

Collecting terms and transposing,

$$-2\sqrt{(2x+5)(x+2)}=-2x-8,$$

or, dividing through by $-2$,

$$\sqrt{(2x+5)(x+2)}=x+4.$$

Squaring again,

$$(2x+5)(x+2)=(x+4)^2,$$

or

$$2x^2+9x+10=x^2+8x+16,$$

or

$$x^2+x=6.$$

Solving,

$$x=2 \text{ and } x=-3.$$

Of these values of $x$, we must retain only those which satisfy the given equation when due regard is taken of the signs of its radicals, as explained at the close of § 18. Thus, with $x=2$, the equation becomes $\sqrt{9}-\sqrt{4}=\sqrt{1}$, or $3-2=1$. This being a true equation, $x=2$ is a root. Again, with $x=-3$, the equation becomes $\sqrt{-1}-\sqrt{-1}=\sqrt{-4}$, or $0=\sqrt{-4}$, which is false. Hence $-3$ is not a root.

## EXERCISES

Solve, by the method shown in § 19, each of the following equations and verify your answer in each case.

**1.** $x-1+\sqrt{x+5}=0$.

**2.** $\sqrt{3x+1}-2\sqrt{2x}=-3$.

**3.** $\sqrt{4x+17}+\sqrt{x+1}-4=0$.

**4.** $\sqrt{2x+1}=2\sqrt{x}-\sqrt{x-3}$.

**5.** $\sqrt{x-a^2}+\sqrt{x+2a^2}=\sqrt{x+7a^2}$.

**6.** $2\sqrt{5x}-\sqrt{2x-1}=\dfrac{4x+1}{\sqrt{2x-1}}$.

**7.** $\sqrt{4x+3}+\sqrt{2x+3}=\sqrt{5x+1}+\sqrt{x+5}$.

**8.** $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}$.

## APPLIED PROBLEMS

**1.** Divide 20 into two parts whose product is 96.

[Hint. Let $x$ be one part. Then $20-x$ will be the other part and we shall have $x(20-x)=96$.]

**2.** Find two consecutive numbers the sum of whose squares is 61.

[Hint. Two numbers are called consecutive when the larger is 1 greater than the smaller.]

**3.** A rectangular garden is 12 rods longer than it is wide and it contains 1 acre. What are its dimensions?

**4.** By increasing each of the edges of a certain cube by 1 inch the volume became increased by 19 cubic inches. What was the original length of each edge?

**5.** A polygon of $n$ sides has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a polygon with 54 diagonals?

**6.** The inner of two concentric circles has a radius of 1 inch. What must be the width of the ring between the circles in order that its area may equal that of the inner circle?

[HINT. The area of a circle is (approximately) 22/7 times the square of its radius.]

**7.** If a train had traveled 6 miles an hour faster it would have required 1 hour less to run 180 miles. How fast did it travel?

[HINT. *Time* = *Distance* ÷ *Rate*.]

**8.** A man can row down stream 16 miles and back in 10 hours. If the stream runs 3 miles an hour, what is his rate of rowing in still water?

**9.** Several persons hired an automobile for $12, but three of them failed to pay their share and as a result each of the others had to advance 20 cents more. How many persons were in the party?

**10.** A cistern is filled by two pipes in 18 minutes; by the greater pipe alone it can be filled in 15 minutes less than by the smaller. Find the time required to fill it by each.

**11.** From three equal sticks are cut off lengths of 7, 8 and 15 inches respectively; the remaining lengths form a right triangle. How long were the sticks?

[HINT. See formula 6, § 7.]

**12.** What is the area of a square whose diagonal is 1 foot longer than a side?

**13.** A rectangle of perimeter 34 inches is inscribed in a circle of diameter 13 inches. Find its sides.

**14.** In order to get from one corner of a rectangular city park to the opposite corner I must go 160 yards round the sides, and of this amount I could save 40 yards if I were allowed to cut diagonally across. What are the dimensions of the park?

**15.** Two airplanes pass over Chicago, one flying east at 40 miles an hour, the other south at 30 miles an hour. The faster machine passes at noon and the other one-half hour later. When are the machines 136 miles apart?

**16.** The formula

$$h = a + vt - 16t^2$$

gives, approximately, the height $h$ of a body at the end of $t$ seconds if it is thrown vertically upwards, starting with a velocity of $v$ feet per second, from a position $a$ feet high.

From the above formula, show that

$$t=\frac{v\pm\sqrt{v^2+64(a-h)}}{32}.$$

Explain the physical meaning of the double sign.

**17.** By means of the result in Ex. 16, find how long it will take a sky-rocket to reach a height of 796 feet if it starts from a platform 12 feet high with an initial velocity of 224 feet per second.

**18.** When a body is thrown vertically downward from a point $a$ feet high and with an initial velocity of $v$ feet per second, its height at the end of $t$ seconds is given by the formula $h=a-vt-16t^2$, which, when solved for $t$ gives

$$t=\frac{-v+\sqrt{v^2+64(a-h)}}{32}. \quad \text{(Compare Ex. 16).}$$

By use of this result, find to the nearest second the time it will take a ball to reach the ground if thrown vertically downward from the top of the Eiffel Tower with an initial velocity of 24 feet per second, the height of the tower being 984 feet.

**19.** A stone is dropped into a well and 4 seconds afterward the report of its striking the water is heard. If the velocity of sound is taken as 1190 feet per second, what is the depth of the well?

[Hint. See Ex. 18.]

**20.** When $s$ feet of wire are stretched between two poles $L$ feet apart (the two points of suspension being regarded as of the same height) the sag $d$ of the wire in feet is given approximately by the formula

$$d=\sqrt{\frac{3Ls-3L^2}{8}}.$$

Solve this formula for $L$ and interpret your answer.

**21.** The whole surface $S$ of a right cylinder of height $h$ and radius $r$ is given by the formula $S=2\pi r(r+h)$. Solve this for $r$ and interpret your answer in words.

**22.** A soap-bubble of radius $r$ is blown out until the area of its outer surface becomes double its original value. Show that the radius has thereby been increased by the amount $r\ (\sqrt{2}-1)$.

[Hint. The area of a sphere whose radius is $r$ is $4\pi r^2$.]

# CHAPTER III

## PROPERTIES OF QUADRATIC EQUATIONS

**20. The Typical Form of Every Quadratic.** We may evidently regard the equation

$$(1) \qquad ax^2+bx+c=0$$

as the *typical form* of every quadratic equation, because every quadratic, being of the second degree, can be brought into the form (1) by a suitable rearrangement of its terms. It is to be here understood that the coefficients $a$, $b$, and $c$ represent numbers which are in no wise dependent upon the unknown number represented by the letter $x$, and that $a$ is not zero, for if it were, (1) would reduce to $bx+c=0$ and hence no longer be an equation of the second degree.

### EXERCISES

Arrange each of the following quadratic equations in the typical form (1) and state the values of $a$, $b$, and $c$ for each.

**1.** $2x^2+5=x(x-1)+7.$

Solution. Transposing all terms to the left, we have the new equation

$$2x^2+5-x^2+x-7=0, \text{ or, combining, } x^2+x-2=0.$$

This is in the form (1) with $a=1$, $b=1$, $c=-2$.

**2.** $3x(x-1)=x^2+2x-1.$

**3.** $4x^2=(x-1)(x+1).$

**4.** $\dfrac{1}{x}-\dfrac{1}{x+1}=2.$

**5.** $(x+m)^2+(x-m)^2=5mx.$

Solution of Ex. 5. We have $x^2+2mx+m^2+x^2-2mx+m^2-5mx=0$, or, combining terms,

$$2x^2-5mx+2m^2=0.$$

This is in the form (1) with $a=2$, $b=-5m$, $c=2m^2$.

**6.** $2x^2+\dfrac{mn}{2}=(m+n)x.$

**7.** $\dfrac{x^2-pq}{x-q}=\dfrac{x+p}{2}.$

**8.** $x^2+(mx+b)^2=r^2.$

**9.** $\dfrac{4k^2}{x+2}-\dfrac{l^2}{x-2}=\dfrac{4k^2-l^2}{x(4-x^2)}.$

**21. Solution of the General Quadratic.** Since the equation

$$ax^2+bx+c=0 \tag{1}$$

is the typical form of every quadratic, it is spoken of as the *general quadratic equation.* We may regard it as a literal equation (§ 7) and solve it by the method of completing the square (§ 12) as follows:

Transposing the term $c$ to the right, then dividing through by $a$ and finally adding $[b/(2a)]^2$ to both members of (1), it becomes

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2. \tag{2}$$

The left member is now a perfect square, namely

$$\left(x+\frac{b}{2a}\right)^2,$$

while the right member readily reduces to

$$\frac{b^2-4ac}{4a^2}.$$

Thus (2) is the same as

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}. \tag{3}$$

By extracting the square root of each member of (3) we obtain

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}. \tag{4}$$

Hence, upon transposing the $b/(2a)$ in (4), it follows that the two values of $x$ (which for convenience we will now call $x_1$ and $x_2$) which satisfy (1) must be

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}. \tag{5}$$

These, then, are the values of the two roots, or solutions, of (1). By means of these formulas, we may at once solve any given quadratic, as illustrated below.

Example. Solve the quadratic $4x^2+8x-5=0$.

Solution. Here $a=4$, $b=8$, $c=-5$. Substituting these values for $a$, $b$ and $c$ in the formulas (5), it appears that the two roots in the present case (when written together in condensed form) are

$$\frac{-8\pm\sqrt{8^2-4(4)(-5)}}{2\cdot 4},$$

which reduces to

$$\frac{-8\pm\sqrt{144}}{8}, \text{ or } \frac{-8\pm 12}{8}.$$

Taking the $+$ sign, this becomes $(-8+12)/8$, which reduces to $1/2$, while if we take the $-$ sign, it becomes $(-8-12)/8$, which reduces to $-5/2$.

The two desired solutions are therefore $1/2$ and $-5/2$.

Check. $4(\frac{1}{2})^2+8(\frac{1}{2})-5=4(\frac{1}{4})+4-5=1+4-5=0$.

Again,

$$4\left(-\frac{5}{2}\right)^2+8\left(-\frac{5}{2}\right)-5=4\left(\frac{25}{4}\right)-8\left(\frac{5}{2}\right)-5=25-20-5=25-25=0.$$

## EXERCISES

Solve each of the following quadratic equations by use of the formulas (5).

**1.** $2x^2+5x+2=0$.

**2.** $3x^2+11x+6=0$.

**3.** $6x^2-7x+2=0$.

**4.** $4x^2+4x-15=0$.

**5.** $3x^2-13x=10$.

**6.** $2x^2+3x=1$.

**7.** $3x^2+2x-4=0$.

**8.** $3x^2-6x=-2$.

**9.** $x^2-6x+10=0$.

**10.** $x^2-mx=mn-nx$.

[Hint. First arrange in the form (1). See § 20.]

**11.** $x^2+mx=mn+nx$.

**12.** $x^2-3bx=2ax-6ab$.

**13.** $x^2+4mx+3nx=-12mn$.

**14.** $x^2-2ax-a^2=0$.

**15.** $6x^2+3ax=2bx+ab$.

**16.** $x^2+px+q=0$.

**17.** By substituting into equation (1) the values of $x_1$ and $x_2$ given in (5), verify the fact that $x_1$ and $x_2$ satisfy (1).

**22. Nature of the Solutions. Discriminant.** If the coefficients $a$, $b$, $c$ in (1), § 21, are real numbers, the form of the two solutions (5) there obtained shows that neither solution can be imaginary unless the expression $b^2-4ac$ has a negative value. In fact, these solutions contain no radicals except $\sqrt{b^2-4ac}$ and this is imaginary only when $b^2-4ac$ has a negative value (§ 10). If, then, the coefficients $a$, $b$, $c$ of any given quadratic (1) are such that $b^2-4ac$ is *positive* the two solutions will be *real*, while if $b^2-4ac$ is *negative* the two solutions will be imaginary.

Moreover, if $b^2-4ac$ is *equal to zero*, the two solutions will be equal to each other, since then $\sqrt{b^2-4ac}$ reduces to zero, so that each of the two roots [see (5)] reduces to the simple expression $-b/(2a)$.

Finally, if $b^2-4ac$ is a perfect square, it is possible to find the exact value of $\sqrt{b^2-4ac}$ and hence, in case $a$, $b$, $c$ are rational numbers, it then follows from (5) that the two roots of the given equation will reduce to rational numbers; while if, on the other hand, $b^2-4ac$ is *not* a perfect square, the two roots of the given equation will not be so reducible and will therefore be irrational.†

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†The precise meaning of the terms rational number, irrational number, real number, and imaginary number, is as follows: A *real* number is one whose expression does not require the square root of a negative quantity, while an *imaginary* number is one whose expression does require such a square root. It can be shown that all numbers of algebra fall into one or the other of these two general classes. Thus $1$, $3$, $-2$, $1/2$, $-2/3$, $\sqrt{2}$, $1+\sqrt{3}$ are all real numbers, while $\sqrt{-2}$, $\sqrt{-1/2}$, $2+\sqrt{-3}$ are all imaginary. Moreover, whenever a real number can be expressed in the particular form $p/q$, where $p$ and $q$ are integers (positive or negative, or zero, except that $q$ must not be zero) it is called a *rational* number, while if it cannot be so expressed it is called an *irrational* number. Thus, $1/2$, $-2/3$, $4/7$, $5$, $73$, $-10$ are rational, while $\sqrt{2}$, $\sqrt{5}$, $\sqrt{2/3}$, $\sqrt[3]{1/2}$, $\sqrt[3]{9}$, $1+\sqrt{6}$, $\pi$ are irrational.

For the definitions of *pure imaginary number* and *complex number* and a study of their properties, see § 10, page 26.

In summary, then, we may state the following rule.

RULE. *For any given quadratic equation* $ax^2+bx+c=0$ *whose coefficients,* $a$, $b$, $c$ *are real numbers, the two roots will be*

(1) *Real and unequal if* $b^2-4ac$ *is positive;*

(2) *Real and equal if* $b^2-4ac=0$, *each root then reducing to* $-b/(2a)$;

(3) *Imaginary if* $b^2-4ac$ *is negative.*

*Moreover, if the coefficients* $a$, $b$, $c$ *are rational numbers, the two roots will be*

(4) *Rational if* $b^2-4ac$ *is a perfect square;*

(5) *Irrational if* $b^2-4ac$ *is not a perfect square.*

Because of the manner in which the nature of the solutions of a quadratic equation thus comes to depend upon the value of $b^2-4ac$, this expression is called the ***discriminant*** of the quadratic equation.

EXAMPLE 1. Determine (without solving) the nature of the roots of the quadratic equation

$$x^2-7x-8=0.$$

SOLUTION. Here $a=1$, $b=-7$, $c=-8$. Hence the discriminant, or $b^2-4ac$, has the value $(-7)^2-4(-8)=49+32=81$, which is *positive*. Therefore, by (1) of the rule, the solutions are real and unequal.

Moreover, since 81 is a perfect square, namely $9^2$, it follows from (4) of the rule that the two solutions are rational.

These results may be checked by actually solving the equation and examining the nature of the solutions thus obtained.

EXAMPLE 2. Determine the nature of the roots of the equation

$$3x^2+2x+1=0.$$

SOLUTION. Here $a=3$, $b=2$, $c=1$. Hence, $b^2-4ac=4-12=-8$. Therefore, by (3) of the rule, the solutions must be imaginary.

EXAMPLE 3. Determine the nature of the solutions of the equation

$$4x^2-4x+1=0.$$

SOLUTION. Here $a=4$, $b=-4$, $c=1$. Hence $b^2-4ac=16-16=0$.

Therefore, by (2) of the rule, the two solutions must be real and equal.

## EXERCISES

Determine (without solving) the nature of the solutions of each of the following quadratic equations.

1. $x^2+5x+6=0$.
2. $x^2-7x-30=0$.
3. $2x^2-3x+2=0$.
4. $2x^2-4x+1=0$.
5. $3x^2-x-10=0$.
6. $x^2-x=1$.
7. $x^2+x=1$.
8. $x^2+x=-1$.
9. $9x^2-6x+1=0$.
10. $4x^2+6x-4=0$.
11. $2x^2-9x+4=0$.
12. $7x^2+3x=0$.
13. $4x^2+16x+7=0$.
14. $9x^2+12x=-4$.

**15.** For what values of $m$ will the roots of the quadratic equation $m^2x^2+10x+1=0$ be equal?

SOLUTION. Here (using the language of §20) $a=m^2$, $b=10$, $c=1$ and hence $b^2-4ac=100-4m^2$. According to § 22, the roots of the given equation will therefore be equal if $m$ be so determined that $100-4m^2=0$, that is, if $m^2=25$. Therefore the desired values of $m$ are $+5$ and $-5$.

**16.** In each of the following quadratic equations, find the value (or values) of $m$ which will render the roots equal, and check your result by actually using this value and solving the resulting equation.

(*a*) $x^2+12x+8m=0$.

(*b*) $(m+1)x^2+mx+m+1=0$.

(*c*) $(2x+m)^2=8x$.

(*d*) $2mx^2+x^2-6mx-6x+6m+1=0$.

**17.** For what values of $k$ will the roots of the following quadratic equation in $x$ be equal?

$$a^2(mx+k)^2+b^2x^2=a^2b^2.$$

### 23. The Sum and Product of the Solutions.

We have seen (§ 21) that the two solutions $x_1$, $x_2$ of any quadratic equation

$$ax^2+bx+c=0$$

are given by the formulas

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

It is now to be observed that if we add these two solutions together, the radical cancels and we obtain the simple result

$$x_1+x_2=\frac{-2b}{2a}=-\frac{b}{a}.$$

Again, if we multiply the two solutions together, the result reduces to a very simple form. Thus

$$x_1 \cdot x_2 = \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

These results for $x_1+x_2$ and $x_1 \cdot x_2$ may be summarized in the following useful rule:

**Rule.** *In the general quadratic equation* $ax^2+bx+c=0$, *the sum of the two solutions is* $-b/a$, *while the product of the two solutions is* $c/a$.

**Example.** State the sum and the product of the solutions of the equation $3x^2-2x+6=0$.

**Solution.** Here $a=3$, $b=-2$, $c=6$. Hence the *sum* of the solutions is $-(-2)/3$, or $2/3$, while their *product* is $6/3=2$.

## EXERCISES

State (by inspection) the sum and the product of the solutions of each of the following equations. Check your answer in Exs. 1, 2, 3, 4 by actually solving these equations and thus obtaining the sum and product of the two solutions.

**1.** $3x^2+6x-1=0$.

**2.** $2x^2-5x+3=0$.

**3.** $x^2-2x+1=0$.

**4.** $5x^2-4x+2=0$.

**5.** $6x^2+7x=42$.

**6.** $x^2+\frac{1}{2}x+\frac{1}{7}=0$.

**7.** $2x^2+\sqrt{3}\,x-\sqrt{5}=0$.

**8.** $x^2+px=q$.

**9.** Show that in the quadratic equation $x^2+mx+n=0$ the sum of the solutions is $-m$ and their product is $n$. This general result may be stated in the following useful rule:

**Rule.** *If in a quadratic equation the coefficient of* $x^2$ *is* 1, *the sum of the solutions will be the coefficient of* $x$ *with its sign changed, while the product of the solutions will be the remaining (last) term.*

Explain and illustrate this in the case of the equation $x^2-10x+12=0$.

**10.** Apply the rule stated in Ex. 9 to determine the sum and the product of the solutions of each of the following quadratic equations.

(*a*) $x^2-4x+3=0$.

(*b*) $x^2+x-1=0$.

(*c*) $x^2-10x+13=0$.

(*d*) $x^2-\frac{1}{2}x+\frac{1}{3}=0$.

(*e*) $x^2-\sqrt{2}\,x+\sqrt{5}=0$.

(*f*) $2x^2-5x+3=0$.

[Hint. First divide through by 2.]

(*g*) $3x^2+\frac{1}{3}x-\sqrt{5}=0$.

**24. Formation of Quadratic Equations Having Given Solutions.** We have seen in Chapter II (also in § 21) how to solve a given quadratic equation, that is, how to determine the two values of the unknown number $x$ which satisfy it. It is frequently desirable to reverse this process, that is, to determine the quadratic equation which has two *given* numbers as its solutions. This can always be done, as is shown below.

Example. Form the quadratic equation whose solutions are $-5$ and 2.

Solution. If $x=-5$, then $x+5=0$. Likewise, if $x=2$, then $x-2=0$. Hence the equation $(x+5)(x-2)=0$, or $x^2+3x-10=0$, will be satisfied when either $x=-5$ or $x=2$. (See § 16.)

The desired equation, whose solutions are $-5$ and 2, is therefore

$$x^2+3x-10=0.$$

This result can be checked, of course, by solving the equation thus found and noting that its solutions turn out to be $-5$ and 2, as desired.

Similarly, if the given values are *any* two numbers $a$ and $b$, the quadratic equation having these values as its solutions is

$$(x-a)(x-b)=0, \qquad \text{or} \qquad x^2-(a+b)x+ab=0.$$

## EXERCISES

Form the quadratic equations whose roots are as follows:

**1.** 1, 2.

**2.** $-1$, $-2$.

**3.** $3, \frac{1}{3}$.

**4.** $-\frac{1}{2}, -\frac{1}{3}$.

**5.** $\sqrt{2}, \sqrt{3}$.

**6.** $\sqrt{8}, -\sqrt{2}$.

**7.** $\frac{1}{2}, \sqrt{5}$.

**8.** $\frac{1}{2}\sqrt{5}, -\frac{1}{2}$.

**9.** $3m, -2m$.

**10.** $(a-b), (a+b)$.

**11.** $2+\sqrt{2}, 2-\sqrt{2}$.

**12.** $2\pm\sqrt{3}$.

**13.** $-\frac{1}{2}(3\pm\sqrt{6})$.

**14.** $\frac{1}{2}(-1\pm\sqrt{2})$.

**15.** Show that in case the quadratic equation $ax^2+bx+c=0$ has one solution double the other, then $2b^2=9ac$.

[Hint. Let $r$ be one solution. Then, from what the problem assumes, the other root will be $2r$. Now form the quadratic having $r$ and $2r$ as its solutions, and examine the coefficients.]

**16.** Show that in case the quadratic equation $ax^2+bx+c=0$ has one solution three times the other, then $16ac=3b^2$.

**25. Graphical Solution of Quadratics.** Consider the quadratic equation

(1) $$x^2-3x-4=0.$$

Let us represent the left member by $y$; that is, let us place

(2) $$y=x^2-3x-4.$$

Now, if we give to $x$ any special value, equation (2) determines a corresponding value for $y$. For example, if $x=0$, then $y=0^2-3\times 0-4=-4$. Again, if $x=1$, then

$$y=1^2-3\times 1-4=-6.$$

The table below shows a number of $x$-values with their corresponding $y$-values determined in this way.

| When $x=$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $-1$ | $-2$ | $-3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| then $y=$ | $-4$ | $-6$ | $-6$ | $-4$ | 0 | 6 | 14 | 0 | 6 | 14 |

The graph of the equation (2) is now obtained by drawing a pair of coordinate axes, as in § 6, then plotting each of the points $(x, y)$ which the table contains, and finally drawing the smooth curve passing through all such points, as in Fig. 13. Observe that this graph is *not* a straight line and hence is essentially different in character from the graph of a linear equation (see § 6.) And it is especially important to note that the graph here cuts the $x$-axis in *two* points whose $x$-values (abscissas) are respectively $-1$ and 4. These special $x$-values, determined in this purely graphical way, are the two *solutions* of the given equation (1), for they are those values of $x$ which make $y=0$, that is, that make $x^2-3x-4=0$.

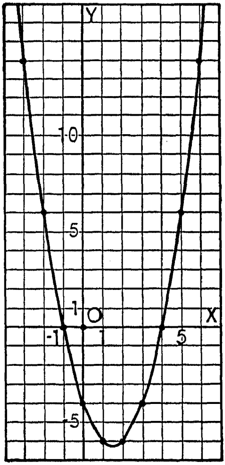

FIG. 13

The graphical study which we have just made for the special equation $x^2-3x-4=0$ leads at once to the following general statements.

*Every quadratic equation has a graph which is obtained by first placing y equal to the left member of the equation* (*it being understood that the right member is* 0), *then letting x take a series of values and determining their corresponding y-values, plotting the points* $(x, y)$ *thus obtained, and finally drawing the smooth curve through them.*

*The x-values of the two points where the graph cuts the x-axis will be the solutions of the given quadratic equation.*

### EXERCISES

Draw the graphs of each of the following equations, and note where each cuts the $x$-axis. In this way determine graphically the values of the solutions, and check the correctness of your answers by actually solving the equation.

**1.** $x^2-x-2=0$.

**2.** $x^2-10x+24=0$.

**3.** $x^2-2x-15=0$.

**4.** $3x^2-8x=3$.

**5.** $2x^2+5x+2=0$.

**6.** $x^2-7x+12=0$.

**7.** $x^2+7x+12=0$.

**8.** $2x^2+3x=9$.

**26. Determining Graphically Whether Solutions Are Real or Imaginary.** In order to apply the method described in § 25 for determining graphically the solutions of a given quadratic it was essential that the graph should cut the $x$-axis. However, quadratic equations may easily be found whose graphs do not cut or touch the $x$-axis at all. For example, consider the equation

$$x^2-6x+15=0. \tag{1}$$

Proceeding as in § 25 to draw the graph, we place

$$y=x^2-6x+15, \tag{2}$$

and determine various pairs of values $(x, y)$ which satisfy this equation. The table below shows several such $(x, y)$ pairs.

| When $x=$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| then $y=$ | 22 | 15 | 10 | 7 | 6 | 7 | 10 | 15 |

Plotting the various points $(x, y)$ thus obtained and drawing the curve through them, we obtain the graph indicated in Fig. 14. It is to be noted that this graph lies entirely *above* the $x$-axis, thus not cutting (or touching it) in any manner. The significance of such a result is that the two roots of (1) are *imaginary*. If they were real, the graph would cut the $x$-axis, as shown in § 25. In reality, we find upon solving (1) that its two solutions have the following imaginary values:

$$x=3\pm\sqrt{-6}.$$

Thus, in general, we have the following result:

*The solutions of a quadratic equation are real or imaginary according as its graph does or does not cut or touch the $x$-axis.*

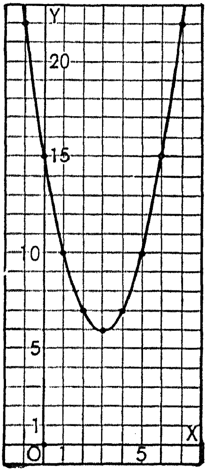

FIG. 14

EXERCISES

Find, by drawing the graph, whether the solutions of each of the following equations are real or imaginary.

**1.** $x^2+2x+3=0.$ **3.** $x^2-2x+3=0.$ **5.** $6x^2+5x+1=0.$

**2.** $x^2+2x-3=0.$ **4.** $3x^2+4x+1=0.$ **6.** $2x^2-3x+4=0.$

**27. The Nature of the Solutions Considered Geometrically.** We have seen in §§ 25, 26 that whenever a quadratic has two distinct real solutions its graph will cut the $x$-axis in two points, while if the solutions are imaginary the graph fails to cut the $x$-axis at all. Suppose now that we have a quadratic equation whose two solutions are real and *equal* to each other, for example the equation

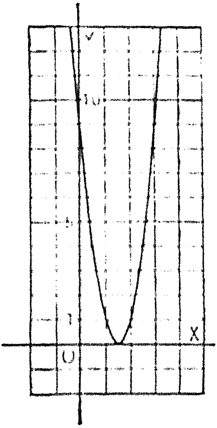

FIG. 15

(1) $$4x^2-12x+9=0.$$

Here the discriminant (§ 22) is equal to

$$(-12)^2-4\times4\times9=144-144=0,$$

so that the roots must be equal by the rule of § 22.

If we now proceed to draw the graph corresponding to (1) in the usual manner by placing $y=4x^2-12x+9$, it appears that the resulting graph just *touches* the $x$-axis instead of actually cutting through it. This was to be expected, since the equality of the roots means that there is but one root, and this, when considered as in § 26, can be possible only when the graph merely touches (is tangent to) the $x$-axis.

Thus, in general, we have the following result. *If the two roots of a quadratic equation are real and equal, the graph of the equation will be tangent to the $x$-axis, and conversely.*

### EXERCISES

Draw the graph of each of the following equations and examine whether they do or do not illustrate the statement at the end of § 27. If not, what statement *is* illustrated (see §§ 25–27)?

**1.** $x^2-2x+1=0.$

**2.** $x^2-6x+12=0.$

**3.** $x^2+6x+12=0.$

**4.** $4x^2+4x+1=0.$

**5.** $x^2-2x-8=0.$

**6.** $3x^2+4x+1=0.$

**7.** $3x^2+4x+2=0.$

**8.** $4x^2-12x+9=0.$

# CHAPTER IV

## SIMULTANEOUS QUADRATIC EQUATIONS

### I. ONE EQUATION LINEAR AND THE OTHER QUADRATIC

**28. Graphical Solution.** In § 6 we have seen how to determine graphically the solution of two simple (first degree) equations each of which contains the two unknown numbers $x$ and $y$. The method consists in drawing the graph of each equation, then observing the $x$ and the $y$ of the point where the two graphs intersect. The particular pair of values $(x, y)$ thus obtained constitutes the solution.

We often meet with a pair of equations similar to those just mentioned except that one (or both) of the equations is *not* of the first degree. For example, consider the pair, or system, of equations

$$x-y=1, \tag{1}$$

$$x^2+y^2=25. \tag{2}$$

In order to solve this pair of equations, that is, to find the particular pair (or pairs) of values $(x, y)$ which will satisfy them both, we may proceed graphically in a manner precisely analogous to that employed in the study of simple equations.

Thus the graph of (1) is found (as in § 6) to be the straight line shown in Fig. 16. In order to draw the graph of (2), we first solve this equation for $y$ in terms of $x$, thus obtaining

$$y=\pm\sqrt{25-x^2}. \tag{3}$$

By giving various values to $x$ in (3), we obtain the $y$-values corresponding to each. The table below shows the $y$-values thus obtained corresponding to $x=0$, $+1$, $+2$, etc., to $x=+5$.

| When $x=$ | 0 | +1 | +2 | +3 | +4 | +5 |
|---|---|---|---|---|---|---|
| then $y=$ | $\pm\sqrt{25}$ | $\pm\sqrt{24}$ | $\pm\sqrt{21}$ | $\pm\sqrt{16}$ | $\pm\sqrt{9}$ | $\pm\sqrt{0}$ |
| $=$ | ±5 | ±4.8 | ±4.5 | ±4 | ±3 | 0 |

Observe that to $x=0$ correspond the *two* values $y=\pm 5$; similarly to $x=1$ correspond the *two* values $y=\pm 4.8$ (approximately), etc.

Moreover, if we assign to $x$ the *negative* value, $x=-1$, we find in the same way that corresponding to it $y$ has the two values, $y=\pm 4.8$. Likewise, for $x=-2$ we find $y=\pm 4.5$, etc., the values of $y$ for any negative value of $x$ being the same each time as for the corresponding positive value of $x$.

Plotting all the points $(x, y)$ thus found and drawing the smooth curve through them, we obtain as the graph the curved line shown in Fig. 16. This curve is a *circle*, as appears when we plot more and more of the points $(x, y)$ pertaining to the equation (3).

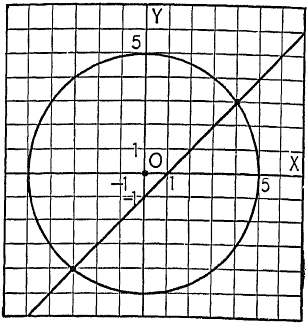

FIG. 16

**NOTE.** The form of (3) shows that there can be no points in the graph having $x$ values greater than 5, for as soon as $x$ exceeds 5 the expression $25-x^2$ becomes *negative* and hence $\sqrt{25-x^2}$ becomes *imaginary*, and there is no point that we can plot corresponding to such a result. Similarly, it appears from (3) that $x$ cannot take values less than $-5$.

Thus the graph can contain no points lying outside the circle already drawn.

Returning now to the problem of solving (1) and (2), we know (§ 6) that wherever the one graph cuts the other we shall have a point whose $x$ and $y$ form a solution of (1) and (2), that is, we shall have a pair of values $(x, y)$ that will satisfy *both* equations at once. From the figure it appears that there are in the present case *two* such points, namely $(x=4, y=3)$ and $(x=-3, y=-4)$. Equations (1) and (2) therefore have the *two* solutions $(x=4, y=3)$ and $(x=-3, y=-4)$. *Ans.*

CHECK. For the solution ($x=4$, $y=3$) we have $x-y=4-3=1$, and $x^2+y^2=16+9=25$, as required.

For the solution ($x=-3$, $y=-4$) we have $x-y=-3-(-4)=1$, and $x^2+y^2=9+16=25$, as required.

The following are other examples of the graphical study of non-linear simultaneous equations.

EXAMPLE 1. Solve the system

$$(4)\qquad 2x-9y+10=0,$$

$$(5)\qquad 4x^2+9y^2=100.$$

SOLUTION. The straight line representing the graph of (4) is drawn readily. To obtain the graph of (5), we have

$$9y^2=100-4x^2.$$

Hence

$$y^2=\tfrac{1}{9}(100-4x^2)=\tfrac{4}{9}(25-x^2),$$

and therefore

$$(6)\qquad y=\pm\tfrac{2}{3}\sqrt{25-x^2}.$$

Corresponding to (6), we find the following table:

| When $x$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | greater than $\pm 5$ |
|---|---|---|---|---|---|---|---|
| then $y$ | $\pm\frac{2}{3}\sqrt{25}$ | $\pm\frac{2}{3}\sqrt{24}$ | $\pm\frac{2}{3}\sqrt{21}$ | $\pm\frac{2}{3}\sqrt{16}$ | $\pm\frac{2}{3}\sqrt{9}$ | $\pm\frac{2}{3}\sqrt{0}$ | imaginary |
| | $\pm\frac{2}{3}(5)$ | $\pm\frac{2}{3}(4.8)$ | $\pm\frac{2}{3}(4.5)$ | $\pm\frac{2}{3}(4)$ | $\pm\frac{2}{3}(3)$ | $\pm 0$ | imaginary |
| | $\pm 3.3$ | $\pm 3.2$ | $\pm 3.0$ | $\pm 2.6$ | $\pm 2$ | 0 | imaginary |

For any negative value of $x$, the $y$-values are the same as for the corresponding positive value of $x$. (See the discussion of (2).)

The graph thus obtained for (6), or (5), is an oval shaped curve. It belongs to the general class of curves called *ellipses*.

The two graphs are seen to intersect at the points

($x=4$, $y=2$) and ($x=-5$, $y=0$).

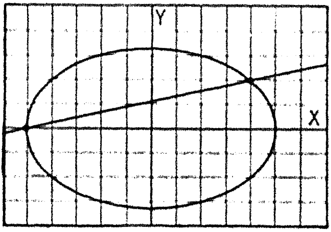

FIG. 17

Therefore the desired solutions of (4) and (5) are ($x=4$, $y=2$) and ($x=-5$, $y=0$). *Ans.*

Example 2. Solve the system

$$2x - y = -2, \tag{7}$$

$$xy = 4. \tag{8}$$

Solution. The graph of (7) is the straight line shown in Fig. 18. To obtain the graph of (8), we have

$$y = \frac{4}{x}, \tag{9}$$

from which we obtain the following table:

| When $x =$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| then $y =$ | $\frac{1}{2}$ | $\frac{4}{7}$ | $\frac{2}{3}$ | $\frac{4}{5}$ | 1 | $\frac{4}{3}$ | 2 | 4 | 8 | 12 | 16 | 20 |

This table concerns only positive values of $x$, but it appears from (9) that for any negative value of $x$ the appropriate $y$-value is the negative of that for the corresponding positive value of $x$.

The graph thus obtained for (9), or (8), consists of two open curves, each indefinitely long, situated as in Fig. 18. These taken together (that is, regarded as one curve) form what is known as a *hyperbola* (pronounced hyper′-bo-la). The part (branch) of the curve lying to the right of the $y$-axis corresponds to the table above, while the other branch corresponds to the negative $x$-values.

The two graphs are seen to intersect in the points ($x = 1$, $y = 4$) and ($x = -2$, $y = -2$).

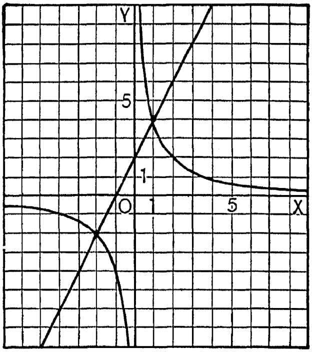

Fig. 18

Therefore the desired solutions of (7) and (8) are ($x = 1$, $y = 4$) and ($x = -2$, $y = -2$). *Ans.*

**Note.** Ellipses and hyperbolas are extensively considered in the branch of mathematics called *analytic geometry*. Both of these curves are of wide application in physics, astronomy, and engineering, as illustrated in the fact that the orbit of each of the planets in the solar system is an ellipse. Both curves belong to a wider class of curves known as the *conic sections*.

Example 3. Consider graphically the system

$$x + y = 10, \tag{10}$$
$$x^2 + y^2 = 25. \tag{11}$$

Solution. The graph of (10) is found in the usual manner, and is represented by the straight line in Fig. 19. The graph of (11) has already been worked out (see discussion of (2)), being a circle of radius 5 with center at the origin. The peculiarity to be especially observed here is that these two graphs *do not intersect*. This means (as it naturally must) that there are no *real* solutions to the system (10) and (11); in other words, the only possible solutions are *imaginary*.

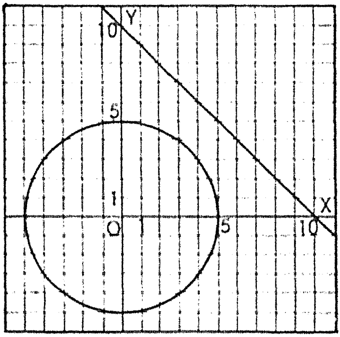

Fig. 19

Likewise, whenever *any* two graphs fail to intersect, we may be assured at once that the only solutions their equations can have are imaginary. The system (10) and (11) and other such systems will be considered further in the next article.

## EXERCISES

Draw the graphs for the following systems and use your result to determine the solutions whenever they are real.

**1.** $\begin{cases} x = 2y, \\ x^2 + y^2 = 20. \end{cases}$

**2.** $\begin{cases} x + y = 7, \\ 3x^2 + y^2 = 43. \end{cases}$

**3.** $\begin{cases} x - 2y = -1, \\ x^2 + 4y^2 = 25. \end{cases}$

**4.** $\begin{cases} x + y = 7, \\ xy = 10. \end{cases}$

**5.** $\begin{cases} x^2 - y^2 = 8, \\ 2x + y = 7. \end{cases}$

**6.** $\begin{cases} x + y = 2, \\ y = x^2. \end{cases}$

**7.** $\begin{cases} 2x + y = 1, \\ y = 4x^2 + 2x + 1. \end{cases}$

**8.** $\begin{cases} x^2 + xy = 12, \\ x - y = 2. \end{cases}$

**9.** $\begin{cases} x = 6 - y, \\ x^3 + y^3 = 72. \end{cases}$

**29. Solution by Elimination.** Let us consider again the system (1) and (2) of § 28.

(1) $$x-y=1,$$

(2) $$x^2+y^2=25.$$

Instead of solving this system graphically, we may solve it by elimination; that is, by the process employed with two linear equations in § 28.

Thus we have from (1)

(3) $$y=x-1.$$

Substituting this value of $y$ in (2), thus eliminating $y$ from (2), we obtain

$$x^2+(x-1)^2=25, \text{ or } x^2+x^2-2x+1=25,$$

or $$2x^2-2x-24=0.$$

or, dividing through by 2,

(4) $$x^2-x-12=0.$$

Solving (4) by formula (§ 56), gives as the two roots

$$x=\frac{-(-1)+\sqrt{(-1)^2-4(1)(-12)}}{2}=\frac{1+\sqrt{1+48}}{2}=\frac{1+7}{2}=4,$$

and

$$x=\frac{-(-1)-\sqrt{(-1)^2-4(1)(-12)}}{2}=\frac{1-\sqrt{1+48}}{2}=\frac{1-7}{2}=-3.$$

When $x$ has the first of these values, namely 4, we see from (3) that $y$ must have the value $y=4-1$, or 3.

Similarly, when $x$ takes on its other value, namely $-3$, we see that $y$ has the value $y=-3-1$, or $-4$.

The solutions of the system (1) and (2) are, therefore, $(x=4, y=3)$ and $(x=-3, y=-4)$. *Ans.*

Observe that these results agree with those obtained graphically for (1) and (2) in § 28.

Further applications of this method are made in the examples that follow.

EXAMPLE 1. Solve the system

$$2x+y=4, \tag{5}$$

$$x^2+y^2=12. \tag{6}$$

SOLUTION. From (5),

$$y=4-2x. \tag{7}$$

Substituting this expression for $y$ in (6), we find

$$x^2+(16-16x+4x^2)=12,$$

or

$$5x^2-16x+4=0. \tag{8}$$

The two roots of (8), as determined by formula (§ 21), are

$$x=\frac{-(-16)\pm\sqrt{(-16)^2-4(5)(4)}}{2(5)}=\frac{16\pm\sqrt{256-80}}{10}=\frac{16\pm\sqrt{176}}{10}$$

$$=\frac{16\pm4\sqrt{11}}{10}\qquad=\frac{8\pm2\sqrt{11}}{5}.$$

The first of these values, namely $x=(8+2\sqrt{11})/5$, when substituted in (7), gives as its corresponding value of $y$,

$$y=4-\frac{16+4\sqrt{11}}{5}=\frac{4-4\sqrt{11}}{5}.$$

The second value, namely $x=(8-2\sqrt{11})/5$, when substituted in (7), gives as its corresponding value of $y$,

$$y=4-\frac{16-4\sqrt{11}}{5}=\frac{4+4\sqrt{11}}{5}.$$

Hence the desired solutions are

$$\begin{cases} x=\dfrac{8+2\sqrt{11}}{5}, \\ y=\dfrac{4-4\sqrt{11}}{5}, \end{cases} \text{and} \quad \begin{cases} x=\dfrac{8-2\sqrt{11}}{5}, \\ y=\dfrac{4+4\sqrt{11}}{5}. \end{cases}$$

To obtain the approximate values of the numbers thus obtained, we have $\sqrt{11}=3.31662$ (tables), and hence the above solutions reduce to the forms

$$\begin{cases} x=2.9266, \\ y=-1.8533, \end{cases} \text{and} \quad \begin{cases} x=0.2734, \\ y=3.4533. \end{cases}$$

These are the solutions of the system (5), (6), *correct to four places of decimals,* which is sufficient for ordinary work.

Example 2. Solve the system

$$x+y=10, \tag{9}$$

$$x^2+y^2=25. \tag{10}$$

Solution. From (9), $y=10-x$. Substituting this expression in (10),

$$x^2+(100-20x+x^2)=25,$$

or

$$2x^2-20x+75=0. \tag{11}$$

Solving (11) by formula, we find its solutions to be, after reduction,

$$x=\frac{10+5\sqrt{-2}}{2} \text{ and } x=\frac{10-5\sqrt{-2}}{2}.$$

Since these $x$-values contain the square root of the negative number $-2$, they are imaginary. The $y$-values are also imaginary, as appears by substituting the $x$-values just found into (9), which gives the results

$$y=\frac{10-5\sqrt{-2}}{2} \text{ and } y=\frac{10+5\sqrt{-2}}{2}.$$

The desired solutions of the systems (9), (10) are therefore

$$\begin{cases} x=\dfrac{10+5\sqrt{-2}}{2}, \\ y=\dfrac{10-5\sqrt{-2}}{2}, \end{cases} \text{ and } \begin{cases} x=\dfrac{10-5\sqrt{-2}}{2}, \\ y=\dfrac{10+5\sqrt{-2}}{2}. \end{cases}$$

This result should now be contrasted with what we saw in Example 3 of § 28 regarding this same system (9) and (10). There we found *graphically* that the solutions must be imaginary because the graphs failed to intersect, but we could not find the actual imaginary numbers which form the solutions.

## EXERCISES

Solve each of the following systems by the method of elimination, and, in case surds are present, find each solution correct to two places of decimals by use of the tables.

**1.** $\begin{cases} x^2+y^2=53, \\ x-y=5. \end{cases}$

**2.** $\begin{cases} 10x+y=3xy, \\ y-x=2. \end{cases}$

**3.** $\begin{cases} x^2+xy=12, \\ x-y=2. \end{cases}$

**4.** $\begin{cases} x-2y=2, \\ x^2+4y^2=25. \end{cases}$

**5.** $\begin{cases} x^2-2y^2=8, \\ x-2y=3. \end{cases}$

**6.** $\begin{cases} x^2+3xy-y^2=43, \\ x+2y=10. \end{cases}$

**7.** $\begin{cases} x^2+3xy=y^2+23, \\ x+3y=9. \end{cases}$

**8.** $\begin{cases} 3x^2-xy-5y^2=5, \\ 3x-5y=1. \end{cases}$

✓ **9.** $\begin{cases} \dfrac{4x}{3y}+\dfrac{2y}{5x}=\dfrac{34}{15}, \\ 2x-5y=-4. \end{cases}$

**10.** $\begin{cases} \dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}=\dfrac{5}{6}, \\ 2x+5y=5. \end{cases}$

## II. NEITHER EQUATION LINEAR

**30. Two Quadratic Equations.** In each of the systems considered in §§ 28, 29 one of the two given equations was linear. However, the same methods of solving may often be employed in case *neither* equation is linear. In such cases *four* solutions may be present instead of two.

EXAMPLE 1. Solve the system

(1) $$9x^2+16y^2=160,$$
(2) $$x^2-y^2=15.$$

SOLUTION. Here only $x^2$ and $y^2$ appear and we begin by finding their values. Thus, multiplying (2) through by 16 and adding the result to (1), we eliminate $y^2$ and find that $25x^2=400$, or

(3) $$x^2=16.$$

Substituting this value of $x^2$ in (2), we find

(4) $$y^2=1.$$

From (3) and (4) we now obtain

(5) $$x=\pm 4 \text{ and } y=\pm 1.$$

Forming all the pairs of values $(x, y)$ that can come from (5), we obtain as our desired solutions

$$(x=4,\ y=1);\ (x=-4,\ y=1);\ (x=4,\ y=-1);$$

and $$(x=-4,\ y=-1).\quad \textit{Ans.}$$

CHECK. Each of these pairs of values of $x$ and $y$ is immediately seen to satisfy both (1) and (2). Let the student thus check each pair.

When considered graphically, equation (1) gives rise to an ellipse (compare § 28, Ex. 1), while (2) gives a hyperbola situated as shown in Fig. 20. These two curves intersect in *four* points which correspond to the four solutions just obtained.

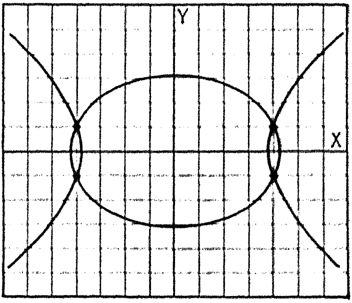

FIG. 20

EXAMPLE 2. Solve the system

(7) $$x^2+y^2=25,$$

(8) $$xy=-12.$$

SOLUTION. Here we cannot proceed as in Example 1 because we cannot find readily the values of $x^2$ and $y^2$. But if we multiply (8) by 2 and add the result to (7), we obtain

(9) $$x^2+2xy+y^2=1.$$

Taking the square root of both members of (9) gives

(10) $$x+y=\pm 1.$$

Similarly, multiplying (8) by 2 and *subtracting* the result from (7),

$$x^2-2xy+y^2=49,$$

and hence

(11) $$x-y=\pm 7.$$

Taking account of the two choices of sign in (10) and (11), we see that they give rise to the four simple (linear) systems:

(*a*) $x+y=1,\ x-y=7$;
(*b*) $x+y=-1,\ x-y=7$;
(*c*) $x+y=1,\ x-y=-7$;
(*d*) $x+y=-1,\ x-y=-7$.

Thus we have replaced the original system (7) and (8) by the four simple systems (*a*), (*b*), (*c*), and (*d*), each of which may be immediately solved by elimination, as in § 28. Since the solutions of (*a*), (*b*), (*c*), (*d*) are respectively ($x=4$, $y=-3$), ($x=3$, $y=-4$), ($x=-3$, $y=4$), and ($x=-4$, $y=3$), we conclude that these are the desired solutions of (7) and (8). *Ans.*

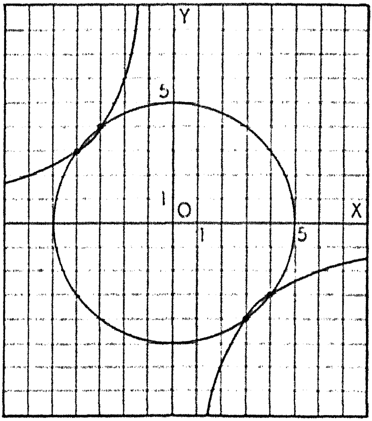

FIG. 21

The graphical significance of these solutions is shown in Fig. 21, where the circle $x^2+y^2=25$ is cut by the hyperbola $xy=-12$ in *four* points that correspond to the four solutions just found.

CHECK. That these four solutions each satisfy (7) and (8) appears at once by trial.

While no general rule can be stated for solving two equations neither of which is linear, the following observation may be made. Unless the equations can be solved readily for $x^2$ and $y^2$ (as in Example 1), the system should first be examined with a view to making such combinations as will yield one or more new systems each of which can be solved (as in Example 2) by methods already familiar. All solutions obtained in this way should be checked in order to avoid false combinations of the $x$- and $y$-values thus obtained.

### EXERCISES

Solve each of the following systems, and draw a diagram for each of the first three to show the geometric meaning of your solutions.

**1.** $\begin{cases} x^2+y^2=10, \\ x^2-y^2=8. \end{cases}$

**2.** $\begin{cases} 4x^2+9y^2=73, \\ 2x^2-y^2=31. \end{cases}$

**3.** $\begin{cases} x^2+y^2=34, \\ xy=15. \end{cases}$

**4.** $\begin{cases} x^2+xy=-6, \\ xy+y^2=15. \end{cases}$

[HINT TO EX. 4. First add, then subtract the two equations, thus showing that the given system is equivalent to two others each of which may be solved as in § 29. Compare Ex. 2, § 30.]

**5.** $\begin{cases} x^2+xy+y^2=151, \\ x^2+y^2=106. \end{cases}$

**6.** $\begin{cases} x^2+xy+y^2=79, \\ x^2-xy+y^2=37. \end{cases}$

**7.** $\begin{cases} xy-6=0, \\ x^2+y^2=xy+7. \end{cases}$

**8.** $\begin{cases} x^2+xy=77, \\ xy-y^2=12. \end{cases}$

**9.** $\begin{cases} s^2-t^2=15, \\ s=4t^2. \end{cases}$

**10.** $\begin{cases} 3xy+2x+y=25, \\ \dfrac{9x}{y}=\dfrac{4y}{x}. \end{cases}$

***31. Systems Having Special Forms.** The systems of equations considered in §§ 29, 30 illustrate the usual and more simple types such as one commonly meets in practice. It is possible, however, to solve more complicated systems provided they are of certain prescribed forms. We shall here consider only two such type forms.

I. *When one (or both) of the given equations is of the form*

$$ax^2+bxy+cy^2=0,$$

*where the coefficients $a$, $b$, $c$ are such that the expression $ax^2+bxy+cy^2$ can be factored into two rational linear factors.*

**Example.** Solve the system

$$x^2+2x-y=7, \tag{1}$$

$$x^2-xy-2y^2=0. \tag{2}$$

**Solution.** Here we see that (2) is of the form mentioned above, since $x^2-xy-2y^2$ can be factored into $(x-2y)(x+y)$. (2) may thus be written in the form

$$(x-2y)(x+y)=0. \tag{3}$$

It follows that either

$$x-2y=0, \text{ or } x+y=0.$$

Hence the system (1), (2) may be replaced by the two following systems:

$$\begin{cases} x^2+2x-y=7, \\ x-2y=0, \end{cases} \quad \text{and} \quad \begin{cases} x^2+2x-y=7, \\ x+y=0. \end{cases}$$

Each of these two systems may now be solved as in § 30, and we thus find that the solutions of the first system are

$$(x=2,\ y=1) \text{ and } (x=-\tfrac{7}{2},\ y=-\tfrac{7}{4}),$$

while the solutions of the second system are

$$\begin{cases} x=\tfrac{1}{2}(-3+\sqrt{37}), \\ y=\tfrac{1}{2}(3-\sqrt{37}), \end{cases}$$

and

$$\begin{cases} x=\tfrac{1}{2}(-3-\sqrt{37}), \\ y=\tfrac{1}{2}(3+\sqrt{37}). \end{cases}$$

The desired solutions of (1) and (2) consist, therefore, of these four solutions just obtained. *Ans.*

II. *When both the given equations are of the form*

$$ax^2+bxy+cy^2=d,$$

*where a, b, c and d have any given values (0 included).*

**Example.** Solve the system

$$x^2-xy+y^2=3, \tag{4}$$

$$x^2+2xy=5. \tag{5}$$

**Solution.** Let $v$ stand for the ratio $x/y$; that is, let us set

$$x=vy. \tag{6}$$

Substituting in (4) and (5), we find,

$$v^2y^2-vy^2+y^2=3. \tag{7}$$

$$v^2y^2+2vy^2=5. \tag{8}$$

Solving (7) and (8) for $y^2$,

$$y^2=\frac{3}{v^2-v+1}, \tag{9}$$

$$y^2=\frac{5}{v^2+2v}. \tag{10}$$

Equating the values of $y^2$ given by (9) and (10),

$$\frac{5}{v^2+2v}=\frac{3}{v^2-v+1}.$$

Clearing of fractions,

$$2v^2-11v+5=0. \tag{11}$$

Solving (11) by formula (§ 21),

$$v=\frac{11\pm\sqrt{121-40}}{4}=\frac{11\pm\sqrt{81}}{4}=\frac{11\pm 9}{4}.$$

Therefore $v=5$, or $v=\frac{1}{2}$. Substituting 5 for $v$ in (9), or (10),

$$y^2=\tfrac{1}{7}.$$

Hence

$$y=+\frac{1}{\sqrt{7}}, \text{ or } -\frac{1}{\sqrt{7}}.$$

Substituting $\frac{1}{2}$ for $v$ in (9) or (10), $y^2=4$. Hence $y=+2$ or $-2$.

The only values that $y$ can have are, therefore, $1/\sqrt{7}$, $-1/\sqrt{7}$, 2, and $-2$.

Since $x=vy$ (see (6) ), the value of $x$ to go with $y=1/\sqrt{7}$ is $x=5(1/\sqrt{7})=5/\sqrt{7}$. Similarly, when $y=-1/\sqrt{7}$ we have

$$x=5(-1/\sqrt{7})=-5/\sqrt{7}.$$

Likewise, when $y=2$ (in which case $v=\frac{1}{2}$, as shown above, then by (6) we have $x=\frac{1}{2}\cdot 2=1$.

Again, when $y=-2$, then $x=\frac{1}{2}(-2)=-1$.

Therefore the only solutions which the system (4), (5) can have are ($x=5/\sqrt{7}$, $y=1/\sqrt{7}$); ($x=-5/\sqrt{7}$, $y=-1/\sqrt{7}$); ($x=1$, $y=2$); ($x=-1$, $y=-2$); and it is easily seen by checking that each of these *is* a solution. *Ans.*

**32. Conclusion.** Every system of equations considered in this chapter has been such that we could solve it by finally solving one or more simple quadratic equations. We have examined only special types, however, and the student should not conclude that all pairs of simultaneous quadratics can be solved so simply. In fact, the solution of simultaneous quadratics *in general* involves a study of equations of higher degree than the second such as considered in Chapter XI.

## MISCELLANEOUS EXERCISES

Solve the following simultaneous quadratics. The star (*) indicates that the exercise depends upon § 31.

**1.** $\begin{cases} x^2+y^2=25, \\ x+y=7. \end{cases}$

**2.** $\begin{cases} x^2+y^2=50, \\ xy=7. \end{cases}$

**3.** $\begin{cases} xy(x-2y)=10, \\ xy=10. \end{cases}$

**4.** $\begin{cases} xy+2x=5, \\ 2xy-y=3. \end{cases}$

**5.** $\begin{cases} xy^2+xy=24, \\ xy^3+x=56. \end{cases}$

**6.** $\begin{cases} x^2-xy=6, \\ x^2-y^2=8. \end{cases}$

**7.** $\begin{cases} x^4-y^4=369, \\ x^2-y^2=9. \end{cases}$

**8.** $\begin{cases} x^2+y^2=100, \\ (x+y)^2=196. \end{cases}$

***9.** $\begin{cases} x^2-7xy+12y^2=0, \\ xy+3y-2x=21. \end{cases}$

***10.** $\begin{cases} x^2+xy+2y^2=11, \\ 2x^2+5y^2=22. \end{cases}$

***11.** $\begin{cases} 2x^2+xy-y^2=0, \\ 2x^2+y=1. \end{cases}$

***12.** $\begin{cases} 2x^2-3y-y^2=8, \\ 6x^2-5xy-6y^2=0. \end{cases}$

***13.** $\begin{cases} xy+2y^2=8, \\ x^2+2xy=12. \end{cases}$

***14.** $\begin{cases} x^2-xy-y^2=20, \\ x^2-3xy+2y^2=8. \end{cases}$

**15.** $\begin{cases} x-2y=2(a+b), \\ xy+2y^2=2b(b-a). \end{cases}$

## APPLIED PROBLEMS

In working the following problems, let $x$ and $y$ represent the two unknown quantities, then form two equations and solve them. If radicals occur, find their approximate values by use of the tables.

**1.** The sum of two numbers is 12, and their product is 32. What are the numbers?

**2.** The sum of two numbers is 82, and the sum of their square roots is 10. What are the numbers?

✓ **3.** A piece of wire 48 inches long is bent into the form of a right triangle whose hypotenuse is 20 inches long. What are the lengths of the sides?

**4.** If it takes 52 rods of fence to inclose a rectangular garden containing 1 acre, what are the length and breadth of the garden?

**5.** If, in the adjoining figure, the combined area of the two circles is $15\frac{5}{7}$ square feet and the distance $CC'$ between centers is 3 feet, what are the lengths of the two radii? $\left(\text{Take } \pi = \frac{22}{7}.\right)$

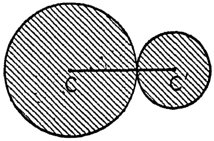

FIG. 22

**6.** Work Ex. 5 in case the circles are situated as in Fig. 23, taking the shaded area to be 110 square feet and $CC'$ to be 5 feet.

**7.** The area of a triangle is 160 square feet, and its altitude is twice as long as its base. Find, correct to three decimal places (using tables), the base and altitude.

**8.** The area of a rectangular lot is 2400 square feet, and the diagonal across it measures 100 feet. Find, correct to three decimal places, the length and breadth.

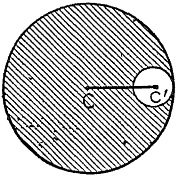

FIG. 23

**9.** The dimensions of a rectangle are 5 feet by 2 feet. Find the amounts (correct to two decimal places) by which each dimension must be changed, and how, in order that both the area and the perimeter shall become doubled.

**10.** Two men working together can complete a piece of work in 6 days. If it would take one man 5 days longer than the other to do the work alone, in how many days can each do it alone? (Compare Ex. 19, p. 12.)

**11.** The fore wheel of a carriage makes 28 revolutions more than the rear wheel in going 560 yards, but if the circumference of each wheel be increased by 2 feet, the difference would be only 20 revolutions. Find the circumference of each wheel.

**12.** A sum of money on interest for one year at a certain rate brought $7.50 interest. If the rate had been 1% less and the principal $25 more, the interest would have remained the same. Find the principal and the rate.

**13.** A man traveled 30 miles. If his rate had been 5 miles more per hour, he could have made the journey in 1 hour less time. Find his time and rate. (See Ex. 10, p. 9.)

**14.** Show that the formulas for the length $l$ and the width $w$ of the rectangle whose perimeter is $a$ and whose area is $b$ are

$$l=\tfrac{1}{4}(a+\sqrt{a^2-16b}),\quad w=\tfrac{1}{4}(a-\sqrt{a^2-16b}).$$

**15.** If the difference of the areas of two circles be $d$ and the sum of their circumferences be $s$, show that their radii $r_1$ and $r_2$, must have the following values:

$$r_1=\frac{4\pi d+s^2}{4\pi s},\quad r_2=\frac{s^2-4\pi d}{4\pi s}.$$

# CHAPTER V

## THE PROGRESSIONS

### I. ARITHMETIC PROGRESSION

**33. Definitions.** An ***arithmetic progression*** is a sequence of numbers, called ***terms,*** each of which is derived from the preceding by adding to it a fixed amount, called the ***common difference.*** An arithmetic progression is commonly denoted by the abbreviation ***A. P.***

Thus 1, 3, 5, 7, $\cdots$ is an A. P., since each term is derived from the preceding by adding 2 to it. Hence 2 is the common difference. The dots following the 7 indicate that the series may be extended as far as one pleases. Thus the first term after 7 would be $7+2$, or 9; the next would be $9+2$, or 11; etc.

Again, 5, 1, $-3$, $-7$, $-11$, $\cdots$ is an A. P. Here the common difference is $-4$.

#### EXERCISES

Determine which of the following progressions are arithmetic progressions, and for such as are, determine the common difference.

**1.** 3, 6, 9, 12, $\cdots$ 3N

**2.** 3, 5, 6, 8, $\cdots$.

**3.** 6, 3, 0, $-3$, $\cdots$.

**4.** 30, 25, 20, 15, $\cdots$.

**5.** $-1$, 2, 5, 8, $\cdots$.

**6.** 0, $2a$, $4a$, $6a$, $\cdots$.

**7.** $a$, $a+4$, $a+8$, $a+12$, $\cdots$.

**8.** $a$, $a+d$, $a+2d$, $a+3d$, $\cdots$.

**9.** $x-4y$, $x-2y$, $x-y$, $\cdots$.

**10.** $3x+3y$, $6x+2y$, $9x+y$, $\cdots$.

**11.** Write the first five terms of the A. P. in which

(*a*) The first term is 4 and the common difference is 2.

(*b*) The first term is $3a$ and the common difference is $-b$.

**34. The Formula for the *n*th Term.** From the definition (§ 33) it follows that every arithmetic progression is of the typical form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots.$$

Here the first term is $a$ and the common difference is $d$.

Observe that the coefficient of $d$ in any given term is 1 less than the number of that term. Thus, in the *third* term the coefficient of $d$ is $3-1$, or 2; likewise in the *fourth* term the coefficient of $d$ is $4-1$, or 3. Thus, in general, the coefficient of $d$ in the $n$th term is $(n-1)$. Hence, if we let $l$ stand for the entire $n$th term, we have the formula

$$l=a+(n-1)d.$$

EXAMPLE. Find the 11th term of the A. P. 1, 3, 5, 7, ⋯.

SOLUTION. Here $a=1$, $d=2$, $n=11$, $l=?$ Hence, substituting in the formula, we find $l=a+(n-1)d=1+10\times2=1+20=21$. *Ans.*

This result may be checked by actually writing out the series so as to include the 11th term.

**35. The Formula for the Sum of the First $n$ Terms.** Let $a$ represent the first term of an A. P., $d$ the common difference and $l$ the $n$th term, as in § 34. Then the sum of the first $n$ terms, which we will denote by $S$, is

(1) $S=a+(a+d)+(a+2d)+(a+3d)+\cdots+(l-d)+l.$

This value for $S$ may be much simplified, however, as we shall now show.

Write the A. P. (1) in its reverse order, thus obtaining

(2) $S=l+(l-d)+(l-2d)+(l-3d)+\cdots+(a+d)+a.$

Now add (1) and (2), noting the cancellation of $d$ with $-d$, of $2d$ with $-2d$, etc. The result is

$$2S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l),$$

or

$$2S=n(a+l).$$

Therefore

$$S=\frac{n}{2}(a+l).$$

If we replace $l$ by its value $a+(n-1)d$ (§ 34), this formula takes the form

$$S=\frac{n}{2}\left\{2a+(n-1)d\right\}.$$

Example. Find the sum of the first 12 terms of the A. P. 2, 6, 14, ···.

Solution. Here $a=2$, $d=4$, $n=12$, $s=?$

Substituting in the second of the formulas just obtained, we find

$$S=\frac{12}{2}\left\{4+11\times 4\right\}=6\left\{4+44\right\}=6\times 48=288. \quad \textit{Ans.}$$

**6. Arithmetic Means.** The terms of an arithmetic pro-ssion that lie between any two given terms are called the *hmetic means* between those terms.

Thus the three arithmetic means between 1 and 9 are 3, 5, 7, since , 5, 7, 9 form an A. P.

Whenever a *single* term is thus inserted between two nbers, it is briefly called the ***arithmetic mean*** of those numbers.

Thus the arithmetic mean of 2 and 10 is 6 because 2, 6, 10 form . P.

A formula for the arithmetic mean between *any* two num-s $a$ and $b$ is easily obtained. Thus, if $x$ is the desired mean, n $a$, $x$, $b$ must form an A. P. Hence, if $d$ be the common erence, we must have $x-a=d$ and $b-x=d$. It follows t we must have $x-a=b-x$. This equation, when solved $x$, gives as the desired formula

$$x=\frac{a+b}{2}.$$

Thus, it follows that *the arithmetic mean of two numbers is al to half their sum.*

Note. The arithmetic mean of two numbers is also called their *age.*

Example. Insert five arithmetic means between 3 and 33.

Solution. We are to have an A. P. of 7 terms in which $a=3$, $l=33$, $n=7$. We begin by finding $d$. Thus

$l=a+(n-1)d$ (§ 34) so that $33=3+6d$. Solving, $d=5$.

The progression is therefore 3, 8, 13, 18, 23, 28, 33, and hence the red means are 8, 13, 18, 23, 28. *Ans.*

## EXERCISES

Find, by the formulas of §§ 34, 35, the numbers called for in Exercises 1–6 below.

**1.** The 12th term of 3, 6, 9, 12, $\cdots$.

**2.** The 21st term of 4, 2, 0, $-2$, $-4$, $\cdots$.

**3.** The 11th term of $x-y$, $2x-2y$, $3x-3y$, $\cdots$.

**4.** The sum of the first ten terms of 3, 6, 9, 12, $\cdots$.

**5.** The sum of the first thirteen terms of 1, $3\frac{1}{2}$, 6, $\cdots$.

**6.** The sum of the A. P. of eleven terms, the first of which is $-5$ and the last of which is 20.

**7.** When a small heavy body (as a bullet) drops vertically downward it passes over 16.1 feet during the first second, three times as far during the second second, five times as far during the third second, etc. Hence answer the following questions.

(*a*) How far does it go during the 12th second?

(*b*) How far does it go during the first twelve seconds?

**8.** If you save 5 cents during the first week in January, 10 cents the second week, 15 cents the third week and so on, how much will you save during the last week of the year. Also, what will be the total of the year's savings?

**9.** Find the sum of all odd integers less than 100.

**10.** The first term of an A. P. is $\frac{1}{2}$ and the 12th term is $11\frac{1}{2}$. What is the sum of the 12 terms?

**11.** In Fig. 24 the sixteen dotted lines are equally spaced, and hence their lengths form an arithmetic progression. If the highest one is 6 inches long and the lowest one is 3 feet long, what is the sum of all their lengths?

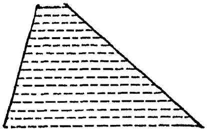

FIG. 24

**12.** The rungs of a ladder diminish uniformly from 2 feet 4 inches in length at the base to 1 foot, 3 inches at the top. If there are 24 rungs altogether, what is the total length of wood they contain?

**13.** A piece of rope, when coiled in the usual manner shown in Fig. 25, is found to have 12 complete turns, or layers. If the innermost turn is 4 inches long and the outermost is 37 inches long, estimate the total length of the rope.

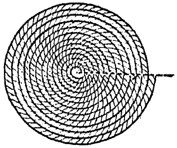

FIG. 25

**14.** Fifty-five logs are to be piled so that the top layer shall contain 1 log, the next layer 2 logs, the next layer 3 logs, etc. How many logs will lie on the bottom layer?

**15.** A row of numbers in arithmetic progression is written down and afterwards all erased except the 7th and the 12th, which are found to be $-10$ and 15 respectively. What was the 20th number?

**16.** A small rope is wound tightly round a cone, as shown in Fig. 26, the number of complete turns being 24. Upon unwinding from the top, the first and second turns are found to measure respectively $2\frac{1}{4}$ inches and $3\frac{1}{4}$ inches. Estimate the length of the rope.

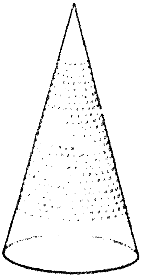

Fig. 26

**17.** Prove that equal multiples of the terms of an arithmetic progression form another arithmetic progression.

**18.** Prove that the sum of $n$ consecutive odd integers, beginning with 1, is $n^2$.

**19.** Show that the first formula for $S$ obtained in § 35 may be translated into words as follows: "The sum of $n$ terms of an arithmetic progression is equal to $n$ multiplied by the arithmetic mean of the first and the last terms."

**20.** In the figure below is shown the frustum of a cone with its "mid-section," or section midway between the bases. Similarly, the frustum of a pyramid and its "mid-section" are shown. It is proved in solid geometry that in all such cases the perimeter of the mid-section is the arithmetic mean of the perimeter of the two bases. Hence, answer the following questions:

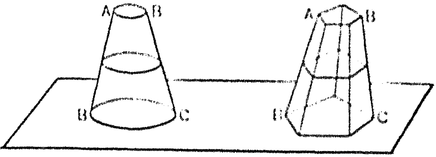

Fig. 27

(*a*) If the perimeters of the bases are 30 inches and 10 inches respectively, what will be the perimeter of the mid-section?

(*b*) If the radius of the upper base is 2 inches and that of the lower base 8 inches, what will be the perimeter of the mid-section?

**21.** If $d=2$, $n=21$ and $S=147$, find $a$ and $l$.

**22.** Show that if any three of the quantities $a$, $d$, $l$, $n$, $S$ are given, it is always possible to find the other two. In particular, prove that the value of $a$ in terms of $d$, $l$ and $S$ is given by the formula

$$a=\tfrac{1}{2}d\pm\sqrt{(l+\tfrac{1}{2}d)^2-2dS}.$$

## II. GEOMETRIC PROGRESSION

**37. Definitions.** A ***geometric progression*** is a sequence of numbers, called ***terms,*** each of which is derived from the preceding by multiplying it by a fixed amount, called the ***common ratio.*** A geometric progression is commonly denoted by the abbreviation ***G. P.***

Thus 2, 4, 8, 16, 32, $\cdots$ is a G. P., since each term is derived from the preceding by multiplying it by 2, which is therefore the common ratio.

Likewise, 10, $-5$, $5/2$, $-5/4$, $\cdots$ is a G. P. whose common ratio is $-1/2$. The next two terms would be $5/8$, $-5/16$.

### EXERCISES

Determine which of the following are geometric progressions, and for such as are, determine the common ratio.

**1.** 3, 6, 12, 24, 48, $\cdots$.

**2.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$.

**3.** $-1, 2, -4, 8, -16, \cdots$.

**4.** $a, a^2, a^3, a^4, \cdots$.

**5.** $(a+b), (a+b)^3, (a+b)^5, (a+b)^7, \cdots$.

**6.** $\frac{m^2}{n^3}, \frac{m^4}{n^4}, \frac{m^6}{n^5}, \frac{m^8}{n^6}, \cdots$.

**7.** Write the first five terms of the G. P. in which

(*a*) The first term is 4 and the common ratio 4.

(*b*) The first term is $-3$ and the common ratio $-2$.

(*c*) The first term is $a$ and the common ratio $r$.

**38. The Formula for the nth Term.** From the definition in § 37 it follows that every geometric progression is of the type form

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots,$$

where $a$ is the first term and $r$ is the common ratio.

Observe that the exponent of $r$ in any one term is 1 less than the number of that term. Thus 2 is the exponent of $r$ in the *third* term; 3 is the exponent of $r$ in the *fourth* term, etc.

Therefore the exponent of $r$ in the $n$th term must be $(n-1)$, so that if we let $l$ stand for the $n$th term we have the formula

$$l=ar^{n-1}.$$

EXAMPLE. Find the 7th term of the G. P. 6, 4, $\frac{8}{3}$, $\cdots$.

SOLUTION. We have $a=6$, $r=\frac{2}{3}$, $n=7$, $l=?$

The formula gives $l=ar^{n-1}=6\times\left(\frac{2}{3}\right)^6=2\times3\times\frac{2^6}{3^6}=\frac{2^7}{3^5}=\frac{128}{243}$. *Ans.*

**39. The Formula for the Sum of the First n Terms.** Let $a$ be the first term of a geometric progression, $r$ the common ratio and $l$ the $n$th term. Then the *sum* of the first $n$ terms, which we will call $S$, is

(1) $$S=a+ar+ar^2+ar^3+\ldots+ar^{n-2}+ar^{n-1}.$$

This value for $S$ may, however, be written in a very much more condensed form, as we shall now show. Multiply both members of (1) by $r$, thus obtaining

(2) $$rS=ar+ar^2+ar^3+ar^4+\ldots ar^{n-1}+ar^n.$$

Now subtract equation (2) from equation (1), noting the cancellation of terms. This gives $S-rS=a-ar^n$. Solving this equation for $S$, we find

(3) $$S=\frac{a-ar^n}{1-r}.$$

This is the condensed form for $S$ mentioned above.

It is to be observed also that since $l=ar^{n-1}$ (§ 38), we may write $rl=ar^n$. Placing this value of $ar^n$ into the formula just found for $S$, we obtain as a second expression for $S$

$$S=\frac{a-rl}{1-r}.$$

EXAMPLE. Find the sum of the first six terms of the G. P. 3, 6, 12, 24, $\cdots$.

SOLUTION. $a=3$, $r=2$, $n=6$, $S=?$

$$S=\frac{a-ar^n}{1-r}=\frac{3-3\cdot2^6}{1-2}=\frac{3-3\cdot64}{-1}=\frac{3-192}{-1}=\frac{-189}{-1}=189.\quad \textit{Ans.}$$

**40. Geometric Means.** The terms of a geometric progression that lie between any two given terms are called the ***geometric means*** between those two terms.

Thus, if we wish to insert three geometric means between 2 and 32, they would be 4, 8, 16, since 2, 4, 8, 16, 32 forms a G. P.

Whenever a *single* term is inserted in this way between two numbers, it is briefly called the ***geometric mean*** of those two numbers.

Thus the geometric mean of 2 and 32 is 8, since 2, 8, 32 forms a G. P.

A formula for the geometric mean of any two numbers, as $a$ and $b$, is easily obtained. Thus, if $x$ denote the mean, then $a$, $x$, $b$ forms a G. P. so that $x/a=b/x$, each of these fractions being equal to the common ratio of the G. P. Clearing this equation of fractions, and solving for $x$ we find

$$x=\sqrt{ab}.$$

Thus it follows that *the geometric mean of two numbers is equal to the square root of their product.*

Example. Insert four geometric means between 3 and 96.

Solution. We are to have a G. P. in which $a=3$, $l=96$ and $n=6$. We begin by finding $r$. Thus

$l=ar^{n-1}$ (§ 38), so that $96=3\cdot r^5$, or $r^5=32$. Hence $r=2$.

The progression is therefore 3, 6, 12, 24, 48, 96, and hence the four desired means are 6, 12, 24, 48. *Ans.*

**Historical Note.** It is related that when Sessa, the inventor of chess, presented his game to Scheran, an Indian prince, the latter asked him to name his reward. Sessa begged that the prince would give him 1 grain of wheat for the first square of the chess board, 2 for the second, 4 for the third, 8 for the fourth, and so on to the sixty-fourth. The number of grains of wheat thus called for was (see (3), § 39)

$$\frac{1-1\cdot 2^{64}}{1-2}=\frac{2^{64}-1}{1}=2^{64}-1=18{,}446{,}744{,}073{,}709{,}551{,}615.$$

This amount is greater than the world's annual supply at present. History does not relate how the claim was settled. (From Godfrey and Siddons' *Elementary Algebra,* Vol. II, pp. 336, 337.)

## EXERCISES

Find, by the formulas of §§ 38, 39, the following numbers.

**1.** The ninth term of 2, 4, 8, 16, ⋯.

**2.** The eighth term of $\frac{1}{4}$, $\frac{1}{2}$, 1, ⋯.

**3.** The tenth term of 4, 2, 1, $\frac{1}{2}$, ⋯.

**4.** The eleventh term of $ax$, $a^2x^2$, $a^3x^3$, $a^4x^4$, ⋯.

**5.** The tenth term of 2, $\sqrt{2}$, 1, ⋯.

**6.** The sum of eight terms of 2, 4, 8, ⋯.

**7.** The sum of six terms of 1, 5, 25, ⋯.

**8.** The sum of ten terms of $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, ⋯.

**9.** The sum of ten terms of 1, $a^2$, $a^4$, ⋯.

**10.** What is the sum of the series 3, 6, 12, ⋯, 384?

**11.** What is the sum of the series 8, 4, 2, ⋯, $\frac{1}{16}$?

**12.** Find the sum of the first ten powers of 2.

**13.** Find the sum of the first seven powers of 3.

**14.** For every person there have lived two parents, four grandparents, eight great grandparents, etc. How many ancestors does a person have belonging to the 7th generation before himself (assuming no duplication)? Answer also for the 10th generation.

**15.** From a grain of corn there grew a stalk which produced an ear of 100 grains. These grains were planted and each produced an ear of 100 grains. This was repeated until there were 5 harvests. If 75 ears make a bushel, how many bushels were there the fifth year?

**16.** A series of five squares is drawn such that a side of the second is twice as long as a side of the first, a side of the third twice as long as a side of the second, etc. If a side of the first is 2 inches long, find (by § 39) the sum of the areas of all the squares.

**17.** Half the air in a certain sealed receptacle is removed by each stroke of an air pump. What fraction of the original amount of air has been removed by the end of the 7th stroke?

**18.** A wheel is making 32 revolutions per second when the steam is turned off and the wheel begins to slow down, making half as many revolutions each second as it did during the preceding second. How long before it will be making only 2 revolutions per second?

**19.** It is found that the number of bacteria in milk doubles every 3 hours. By how much will it be multiplied by the end of one day?

**20.** Show that if a principal of $\$p$ be invested at $r\%$ compound interest, the sums of money accumulating at the ends of successive years will form a geometric progression, but if the investment be made at simple interest, the sums similarly accumulating will form an arithmetic progression.

**21.** From a cask of vinegar $\frac{1}{3}$ the contents is drawn off and the cask then filled by pouring in water. Show that if this is done 6 times, the cask will then contain more than 90% water.

[HINT. Call the original amount of vinegar 1, then express (as a proper fraction) the amount of water in the cask after the first refilling, second refilling, etc.]

**22.** In Fig. 28 a series of ordinates equally spaced from each other has been drawn, the first one being laid off 1 unit long, the second one being laid off equal to the first one increased by $\frac{1}{4}$ its length, etc. Show that these ordinates represent the successive terms of the G. P. whose first term is 1 and whose common ratio is $1\frac{1}{4}$. In this sense, the figure may be called the diagram corresponding to the G. P. in which $a=1$, $r=1\frac{1}{4}$.

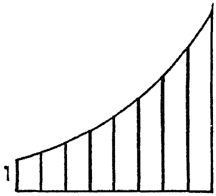

FIG. 28

**23.** Draw the diagram for the G. P. in which

$$(a)\ a=1,\ r=1\tfrac{1}{3},\quad (b)\ a=2,\ r=1\tfrac{1}{6},\quad (c)\ a=4,\ r=\tfrac{1}{2}.$$

**24.** Prove that the reciprocals of the terms of a geometric progression form another such progression.

**25.** If a series of numbers are in geometric progression, are their squares likewise in geometric progression? Answer the same question for the cubes of the given numbers; also for their square roots and their cube roots.

Answer the same questions for an arithmetic progression.

[HINT. See that your reasoning is general; that is, do not base it merely upon the examination of special cases.]

**26.** Find, correct to four decimal places, the geometric mean of 6 and 27. (Use the tables.)

**27.** In Fig. 29 a square is placed (in any manner) within another square whose side is twice as long. Show that the area between the squares is equal to three halves of the geometric mean of the areas of the two squares.

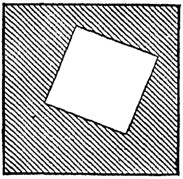

FIG. 29

**41. Infinite Geometric Progression.** Consider the geometric progression

(1) $$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots.$$

Here $a=1$, $r=\frac{1}{2}$, and hence, by § 39, the sum of $n$ terms is

$$S=\frac{a-ar^n}{1-r}=\frac{1-1\cdot(\frac{1}{2})^n}{1-\frac{1}{2}}=\frac{1-(\frac{1}{2})^n}{\frac{1}{2}}.$$

Now, if the value selected for $n$ is very large, the expression $(1/2)^n$ which here appears is very small, being the fraction $\frac{1}{2}$ multiplied into itself $n$ times. In fact, as $n$ is selected larger and larger, this expression $(1/2)^n$ comes to be as small as we please, so that the value for $S$, as given above, comes as near as we please to

$$\frac{1-0}{\frac{1}{2}},$$

which is the same as 2. So we say that 2 is the *sum to infinity* of the geometric progression above, meaning thereby simply that as we sum up the terms, taking more and more of them, we come and remain *as near as we please* to 2.

The meaning of this result is illustrated in Fig. 30.

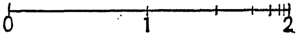

FIG. 30

Here, beginning at the point marked 0, we first measure off 1 unit of length, then, continuing to the right, we measure off $\frac{1}{2}$ unit, then $\frac{1}{4}$ unit, then $\frac{1}{8}$ unit, etc., each time going to the right just one-half the amount we went the time before. As this is kept up *indefinitely*, we evidently come as near as we please to the point marked 2, which is 2 units from 0. This corresponds exactly to what we are doing when we add more and more of the terms of the given progression

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots.$$

A progression like the one just considered, in which the value of $n$ is not stated but may be taken as large as one pleases, is called an *infinite* geometric progression.

Having thus considered the sum to infinity of the special infinite geometric progression (1), let us now suppose that we have *any* infinite geometric progression, as

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,$$

and (as before) that $r$ has some value numerically less than 1. Then the sum of the first $n$ terms is, by § 39

$$S=\frac{a-ar^n}{1-r},$$

and, as $n$ is taken larger and larger, the expression $r^n$ which appears here becomes as small as we please, since we have supposed $r$ to be less than 1. Hence, as $n$ increases indefinitely, the value of $S$ comes as near as we please to

$$\frac{a-a\cdot 0}{1-r},$$

or

$$\frac{a}{1-r}.$$

We have therefore the following theorem: *The sum to infinity of any geometric progression whose common ratio* $r$ *is numerically less than* 1 *is given by the formula*

$$\boldsymbol{S=\frac{a}{1-r}}.$$

Example. Find the sum to infinity of the progression

$$3,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{9},\ \tfrac{1}{27},\ \cdots.$$

Solution. $a=3$, $r=\frac{1}{3}$. Since $r$ is numerically less than 1, we have by the formula of § 41,

$$S=\frac{a}{1-r}=\frac{3}{1-\frac{1}{3}}=\frac{3}{\frac{2}{3}}=\frac{9}{2}=4\tfrac{1}{2}.\quad \textit{Ans.}$$

## EXERCISES

Find the sum to infinity of each of the following progressions, and state in each case what your answer *means*, drawing a diagram similar to Fig. 30 to illustrate.

**1.** $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$.

**2.** $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \cdots$.

**3.** $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \cdots$.

[HINT. $r = -\frac{1}{3}$ and hence is numerically less than 1. The formula of § 41 therefore applies.]

**4.** $4, .4, .04, .004, \cdots$.

**5.** $\frac{1}{8}, -\frac{1}{18}, \frac{2}{81}, \cdots$.

**6.** $1 - x + x^2 - x^3 + \cdots$ when $x = \frac{3}{4}$.

**7.** $\sqrt{3}, 1, \dfrac{1}{\sqrt{3}}, \dfrac{1}{3}, \cdots$.

**8.** $\dfrac{2}{3}, -\dfrac{2\sqrt{2}}{3\sqrt{3}}, \dfrac{4}{9}, \cdots$.

**9.** $\dfrac{4}{5}, \dfrac{2}{5\sqrt{3}}, \dfrac{1}{15}, \cdots$.

**10.** A pendulum starts at $A$ and swings to $B$, then it swings back as far as $C$, then forward as far as $D$, etc. If the first swing (that is, the circular arc from $A$ to $B$) is 6 inches long and each succeeding swing is five-sixths as long as the one just preceding it, how far will the pendulum bob travel before coming to rest?

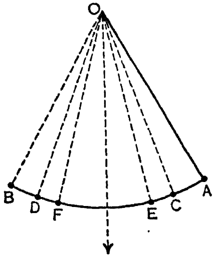

FIG. 31

**11.** At what time after 3 o'clock do the hands of a watch pass each other?

[HINT. We may look at this as follows: The large (minute) hand first moves down to where the small (hour) hand is at the beginning, that is, through 15 of the minute spaces along the dial. Meanwhile the small hand advances $\frac{1}{12}$ as far, or $\frac{15}{12}$ of a minute space. This brings the small hand to the position indicated by the dotted line in the figure. The large hand next passes over this $\frac{15}{12}$ of a minute space. Meanwhile the small hand again advances $\frac{1}{12}$ as far, which is $\frac{15}{144}$ of a minute space. The large hand next covers this $\frac{15}{144}$ of a minute space, but the small hand meanwhile advances $\frac{1}{12}$ as far, or $\frac{15}{1728}$ of a minute space, etc. Thus, the successive moves of the large hand, counting from the first one, form the G. P. $15, \frac{15}{12}, \frac{15}{144}, \frac{15}{1728}, \cdots$.]

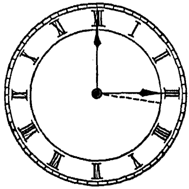

FIG. 32

**42. Variable. Limit.** We have seen (§ 41) in connection with the geometric progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, that the sum of its first $n$ terms is a quantity which, as $n$ increases indefinitely, comes and remains as near as we please to the exact value 2. The usual way of stating this is to say that *as $n$ increases, the sum of the first $n$ terms approaches* 2 *as a* ***limit.*** The sum of the first $n$ terms is here called a ***variable*** since it varies, or changes, in the discussion. A similar remark applies to all the infinite geometric progressions which we have considered. In every case the sum to infinity is the limit which the sum of the first $n$ terms, considered as a variable quantity, is approaching.

NOTE. It may be asked whether the sum of the first $n$ terms of the G. P. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ could ever actually reach its limit 2. The answer is that it may or it may not, depending upon circumstances. Thus, if we think of the terms, beginning with the second, as being added on at the rate of one a minute we could never reach the end of the adding process, since the number of the terms is inexhaustible and hence the minutes required would have no end. In other words, the sum of the first $n$ terms could never reach its limit on this plan. But suppose that instead of this we were to add on the terms with increasing speed as we went forward. For example, suppose we added on the $\frac{1}{2}$ in $\frac{1}{2}$ a minute, then the $\frac{1}{4}$ in $\frac{1}{4}$ of a minute, then the $\frac{1}{8}$ in $\frac{1}{8}$ of a minute, etc. On this plan we would actually reach the limit 2 in 2 minutes of time. Here the constantly increasing speed of the adding process exactly counterbalances the fact that we have an indefinitely large number of terms to add, with the result that we reach the end of the process in the definite time of 2 minutes. This idea is practically illustrated in Ex. 11, p. 82, where the hands of the watch would never pass each other at all except for the fact that the successive moves of the large hand, which constitute the terms of the progression $15, \frac{15}{12}, \frac{15}{144}, \frac{15}{1728}, \cdots$ are added on in less and less time as the process goes on, each being added on in $\frac{1}{12}$ the time occupied by the one just before it.

The question of whether a variable can reach its limit is intimately connected with the famous problem considered by the Schoolmen of antiquity and known as the problem of Achilles and the tortoise. In this problem, Achilles, who is a celebrated runner and athlete, starts out from some point, as $A$, to overtake a tortoise which is at some point, as $T$, the tortoise being famous for the slow rate at which it crawls along. Both start at the same instant and go in the same direction, as indicated in the figure. Achilles soon arrives at the point $T$, from which the tortoise started, but in the meantime the tortoise has gone

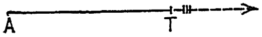

FIG. 33

some distance ahead. Achilles now covers this last distance, but this leaves the tortoise still ahead, having again gained some additional distance. This continues indefinitely. *How, therefore, can Achilles ever overtake the tortoise?* The Schoolmen never quite answered this question satisfactorily to themselves. The secret of the difficulty lies in the fact that, as in the other problems mentioned above, the successive moves which Achilles makes are done in shorter and shorter intervals of *time*, with the result that, although the number of moves necessary is indefinitely great, they can all be accomplished in a definite time.

**43. Repeating Decimals.** If we express the fraction $\frac{12}{33}$ decimally by dividing 12 by 33 in the usual way, we find that the quotient is .363636 ···, the dots indicating that the division process never stops (or is never exact) but leads to a never-ending decimal. However, the digits appearing in this decimal are seen to repeat themselves in a regular order, since they are made up of 36 repeated again and again. Such a decimal is called a *repeating decimal.* More generally, a repeating decimal is one in which the figures repeat themselves *after a certain point.* Thus,

$$.12343434 \cdots, \text{ and } 1.653653653 \cdots,$$

are repeating decimals.

Let us now turn the question around. Thus, suppose that a certain repeating decimal is given, as for example .272727 ···, and let us ask what fraction when divided out gives this decimal. This kind of question is usually too difficult to answer in arithmetic, but it can be easily answered as follows by use of the formula in § 41.

Thus the decimal .272727 ··· may be written in the form

$$\tfrac{27}{100} + \tfrac{27}{10000} + \tfrac{27}{1000000} + \cdots.$$

This is an infinite geometric progression in which $a = \frac{27}{100}$, $r = \frac{1}{100}$. The sum of this progression to infinity must be the value of the given decimal. Hence, the desired value is

$$\frac{a}{1-r} = \frac{\frac{27}{100}}{1 - \frac{1}{100}} = \frac{27}{100} \times \frac{100}{99} = \frac{27}{99} = \frac{3}{11}. \quad \textit{Ans.}$$

This answer may be checked by dividing 3 by 11, the result being .272727 ···, which is the given decimal.

**Note.** It is shown in higher mathematics that every rational fraction in its lowest terms (that is, every number of the form $a/b$, where $a$ and $b$ are integers prime to each other) gives rise when divided out to a never-ending *repeating* decimal (including the cases in which all the digits after a certain point are zero), while every irrational number (such as $\sqrt{2}$) gives rise when expressed decimally to a never-ending *non-repeating* decimal.

### EXERCISES

Find the values of the following repeating decimals and check your answer for each of the first six.

**1.** 0.153153 ···. **2.** 0.135135 ···. **3.** 0.543543543 ···.

**4.** 0.3414141 ···.

Solution.
$$\begin{aligned} 0.3414141 \cdots &= .3 + .0414141 \cdots \\ &= .3 + \tfrac{1}{10}(.414141 \cdots) \\ &= .3 + \frac{1}{10}\left(\frac{\frac{41}{100}}{1 - \frac{1}{100}}\right) \\ &= \frac{3}{10} + \frac{1}{10} \times \frac{41}{100} \times \frac{100}{99} \\ &= \frac{3}{10} + \frac{41}{990} = \frac{338}{990} = \frac{169}{495}. \quad \textit{Ans.} \end{aligned}$$

**5.** 0.17272 ···.

**6.** 1.212121 ···.

**7.** 3.2151515 ···.

**8.** 5.032032032 ···.

**9.** 6.008008008 ···.

**10.** 34.5767676 ···.

# CHAPTER VI

## VARIATION

**44. Direct Variation.** One quantity is said to *vary directly as* another when the two are so related that, though the quantities themselves may change, their *ratio* never changes.

Thus the amount of work a man does varies directly as the number of hours he works. For example, if it takes him 4 hours to draw 10 loads of sand, we can say it will take him 8 hours to draw 20 loads. Here the first ratio is $\frac{4}{10}$ and the second is $\frac{8}{20}$ and the two are equal, though the numbers in the second have been changed from what they were in the first. In general, if the man works twice as long, he will draw twice as much; if he works three times as long, he will draw three times as much, etc.; all of which implies that the *ratio* of the time he works to the amount he draws in that time never changes.

### EXERCISES

Determine which of the following statements are true and which are false, giving your reason in each instance.

**1.** The amount of electricity used in lighting a room varies directly as the number of lights turned on.

**2.** The amount of water in a cylindrical pail varies directly as the height to which the water stands in the pail.

**3.** The amount of gasoline used by an automobile in any given time (one week, say) varies directly as the amount of driving done.

**4.** The time it takes to walk from one place to another at any given rate (3 miles an hour, say) varies directly as the distance between the two places.

**5.** The time it takes to walk any given distance (5 miles, say) varies directly as the rate of walking.

**6.** The perimeter of a square varies directly as the length of one side.

**7.** The circumference of a circle varies directly as the length of the radius.

**8.** The area of a square varies directly as the length of one side.

**9.** $x$ varies directly as $10x$.

**10.** $x$ varies directly as $10x^2$.

**45. Inverse Variation.** One quantity, or number, is said to ***vary inversely as*** another when the two are so related that, though the quantities themselves may change, their *product* never changes.

Thus the time occupied in doing any given piece of work varies inversely as the number of men employed to do it. For example, if it takes 2 men 6 days, it will take 4 men only 3 days. The point to be observed here is that the first product, $2\times 6$, equals the second product, $4\times 3$. In general, if twice as many men are employed it will take *half* as long; if three times as many men are employed, it will take *one-third* as long, etc. In all these cases, the number of men employed multiplied by the corresponding time required to do the work remains the same.

NOTE. The term *varies inversely as* is due to the fact that in case $xy$ never changes (as required by the above definition), it follows that $x \div (1/y)$ never changes, since $xy = x \div (1/y)$. That is, $x$ varies directly as the reciprocal, or *inverse*, of $y$ (§ 44).

### EXERCISES

Determine which of the following statements are true and which are false, giving your reason in each instance.

**1.** The time it takes water to drain off a roof varies inversely as the number of (equal sized) conductor pipes.

**2.** The time it takes to walk any given distance (5 miles, say) varies inversely as the rate of walking.

**3.** The weight of a pail of water varies inversely as the amount of water that has been poured out of it.

**4.** $x$ varies inversely as $10/x$.

**5.** $x$ varies inversely as $10/x^2$.

**46. Joint Variation.** One quantity, or number, is said to ***vary jointly as*** two others when it varies directly as their product.

Thus the area of a triangle varies jointly as its base and altitude, for if $A$ be the area of any triangle and $b$ its base and $h$ its altitude, we have $A = \frac{1}{2}bh$, which may be written $A/bh = \frac{1}{2}$. Hence $A$ varies directly as the product $bh$ (§ 44); that is, the ratio of $A$ to $bh$ is always the same, namely $\frac{1}{2}$ in this instance.

### EXERCISES

Determine whether the following statements are true, giving your reason in each instance.

**1.** The area of a rectangle varies jointly as its two dimensions; that is, as its length and breadth.

**2.** The pay received by a workman varies jointly as his daily wage and the number of days he works.

**3.** The amount of reading matter in a book varies jointly as the thickness of the book and the distance between the lines of print on the page.

**4.** The interest received in one year from an investment varies jointly as the principal and rate.

**5.** The volume of a rectangular parallelopiped (such as an ordinary rectangular shaped box) varies jointly as its length, breadth, and height.

[HINT. Here we have one quantity varying jointly as *three* others. First make a definition of what such variation means.]

**47. Variables and Constants.** When we say that the amount of work a man does varies directly as the number of hours he works, we are dealing with two quantities, namely the amount of work done and the time used in doing it. But it is to be observed that these are not being regarded as fixed quantities, but rather as changeable ones, the only essential idea being that their *ratio* never changes. In general, quantities which are thus changeable throughout any discussion or problem are called ***variables,*** while quantities which do not change are called ***constants.*** (Compare § 42.)

**48. The Different Types of Variation Stated as Equations.** We may now state very briefly and concisely what is meant by the different types of variation described in §§ 44–46 and certain other important types also. To do this, let us think of $x$, $y$, and $z$ as being certain variables and $k$ as being some constant. Then we may state the following facts:

(1) *To say that $x$ varies directly as $y$ means* (§ 44) *that*

$$\frac{x}{y}=k, \text{ or } x=ky, \textit{ where } k \textit{ is a constant.}$$

(2) *To say that $x$ varies inversely as $y$ means* (§ 45) *that*

$$xy=k, \text{ or } x=\frac{k}{y}, \textit{ where } k \textit{ is a constant.}$$

(3) *To say that $x$ varies jointly as $y$ and $z$ means* (§ 46) *that*

$$\frac{x}{yz}=k, \text{ or } x=kyz, \textit{ where } k \textit{ is a constant.}$$

Two other important types of variation are described below:

(4) *To say that $x$ varies directly as the square of $y$ means that*

$$\frac{x}{y^2}=k, \text{ or } x=ky^2, \textit{ where } k \textit{ is a constant.}$$

(5) *To say that $x$ varies inversely as the square of $y$ means that*

$$xy^2=k, \text{ or } x=\frac{k}{y^2}, \textit{ where } k \textit{ is a constant.}$$

In all these types of variation it is important to observe that the value which must be given to the constant $k$ depends upon the particular statement or problem in hand. For example, consider the statement that "The area of a rectangle varies jointly as its two dimensions." This means (see [3]) that if we let $A$ be the variable area and $a$ and $b$ the variable dimensions, then $A=kab$. But in this case we know by arithmetic that $A=ab$, so the value of $k$ here must be 1.

On the other hand, consider the statement that "The area of a triangle varies jointly as its base and altitude." Letting $A$ be the variable area and $b$ and $h$ the variable base and altitude, respectively, this means that $A=kbh$. But here, as we know from geometry, $k=\frac{1}{2}$.

### EXERCISES

Convert each of the following statements into equations, sup for each the proper value for the constant $k$ mentioned in § 48.

**1.** The circumference of a circle varies directly as the radius.

[HINT. Let $C$ stand for circumference and $r$ for radius.]

**2.** The circumference of a circle varies directly as the diam

**3.** The area of a circle varies directly as the square of the r

**4.** The area of a circle varies directly as the square of the di

**5.** The area of a sphere varies directly as the square of the r

**6.** The volume of a rectangular parallelopiped varies jointly length, breadth, and height.

**7.** Interest varies jointly as the principal, rate, and time.

**8.** The volume of a sphere varies directly as the cube of the r

[HINT. First supply for yourself the definition of what this t variation means.]

**9.** The volume of a circular cone varies jointly as the altitud the square of the radius of the base. (See formula (11), § 7).

**10.** The distance, measured in feet, through which a body f dropped vertically downward from a position of rest (as from a wi ledge) varies directly as the square of the number of seconds it been falling.

[HINT. It is found by experiments in physics that the value o constant $k$ is in this case 16 (approximately).]

**11.** The following, like Ex. 10, are statements of well-known p ical laws. Convert each into an equation without, however, atten to supply the proper value of $k$, since to do so requires a study of ph and experiments in laboratories.

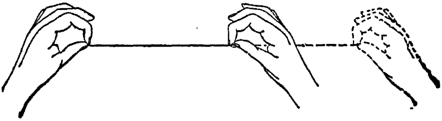

FIG. 34

(*a*) When an elastic string is stretched out, as represented in Fig the tension (force tending to pull it apart at any point) varies dir as the amount of the extension beyond the natural length of the st (*Hooke's Law*) (*b*) If a body is tied to a string and swung round and round in a cle (as in swinging a pail of water at arm's length from the shoulder), force, *F*, with which it pulls outward from the center (called *cen-ugal force*) varies directly as the square of the velocity of the motion.

(*c*) The intensity of the illumination due to any small source of ht (such as a candle) varies inversely as the square of the distance of object illuminated from the source of light.

(*d*) The pressure per square inch which a given amount of gas (such air, or hydrogen, or oxygen, or illuminating gas) exerts upon the les of the containing receptacle (such as a bag) varies inversely as volume of the receptacle (*Boyle's Law*).

For example, whenever air is confined in a rubber balloon, as in e first drawing in Fig. 35, it exerts a certain pressure upon each square

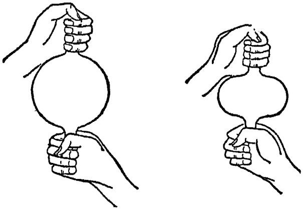

Fig. 35

ch of the interior surface. If the balloon be squeezed, as in the second rawing (no air being allowed to escape), until its volume is half of hat it was before, this pressure will be exactly doubled.

(*e*) The cube of the mean distance of any planet in the solar sys-em from the sun varies directly as the square of the time it takes the lanet to make one complete revolution around the sun (*Kepler's third w of planetary motion*).

In the case of the earth, its mean distance from the sun is about 3,000,000 miles and its time of complete revolution is 1 year, or 55¼ days.

**49. Problems in Variation.** The problems naturally rising in the study of variation fall into two general classes s follows:

(1) Those in which the value of the constant *k* mentioned n § 48 can be determined from the statement of the problem

and forms an essential part in the solution. This kind of problem is illustrated by Exs. 1–10 below. The solution given for Ex. 1 should be well understood before the student undertakes Exs. 2–10.

(2) Those in which it is *not* necessary to know the value of $k$. Such problems are illustrated in Exs. 11–20 below.

The pupil is advised to work several problems from each group rather than to confine his attention to either.

## EXERCISES

### I. ILLUSTRATIONS OF CASE (1)

**1.** In a fleet of ships all made from the same model (that is, of the same shape, but of different sizes) the area of the deck varies directly as the square of the length of the ship. If the ship whose length is 200 feet has 5000 square feet of deck, how many square feet in the deck of the ship which is 300 feet long?

SOLUTION. Let $A$ represent the area of deck on the ship whose length is $l$. Then the given law of variation, expressed as an equation (§ 48), is

(1) $$A = kl^2. \quad (k = \text{some constant.})$$

Since the ship which is 200 feet long has 5000 square feet of deck, it follows from (1) that we must have

$$5000 = k(200)^2.$$

This equation tells us that the value of $k$ in the present problem must be

$$k = \frac{5000}{(200)^2} = \frac{5000}{200 \times 200} = \frac{1}{8}.$$

Placing this value of $k$ in (1), gives us an equation which determines *completely* the relation between $A$ and $l$ in the present problem; that is,

(2) $$A = \tfrac{1}{8}l^2.$$

Now the problem asks how many square feet of deck there are in the ship whose length is 300 feet. This can be found by simply placing $l = 300$ in (2) and solving for $A$. Thus

$$A = \frac{1}{8} \times (300)^2 = \frac{300 \times 300}{8} = 11{,}250 \text{ square feet. } \textit{Ans.}$$

**Note.** Observe that the first step in the above solution is to express *as an equation* the law of variation belonging to the problem. Next, the constant $k$ is determined. After this, the first equation is rewritten in its more exact form obtained by assigning to $k$ its value. The answer is then readily obtained.

These steps should be followed in working each of the Exs. 2–10 which follow.

**2.** In a fleet of ships all of the same model, the ship whose length is 200 feet contains 6000 square feet in its deck. How long must a similar ship be made if its deck is to contain 13,500 square feet?

**3.** To make a suit of clothes for a man who is 5 feet 8 inches high requires 6 square yards of cloth. How much cloth will be required to make a suit for a man of similar build, whose height is 6 feet 2 inches?

[Hint. The areas of any two similar figures vary directly as the *squares* of their heights.]

**4.** If 10 men can do a piece of work in 20 days, how long will it take 25 men to do it?

[Hint. The time required varies inversely as the number of men employed.]

**5.** The horse-power required to propel a ship varies directly as the cube of the speed. If the horse-power is 2000 at a speed of 10 knots, what will it be at a speed of 15 knots?

**6.** A silver loving-cup (such as is sometimes given as a prize in athletic contests) is to be made, and a model is first prepared out of wood. The model is 8 inches high and weighs 12 ounces. What will the loving-cup cost if made 10 inches high, it being given that silver is 17 times as heavy as wood and costs $2.20 an ounce?

[Hint. The volumes and hence the weights of any two similar figures of like material vary directly as the *cubes* of their heights.]

**7.** When electricity flows through a wire, the wire offers a certain resistance to its passage. The unit of this resistance is called the *ohm*, and for a given length of wire the resistance varies inversely as the square of the diameter. If a certain length of wire whose diameter is $\frac{1}{4}$ inch offers a resistance of 3 ohms, what will be the resistance of a similar wire (same length and material) $\frac{1}{3}$ of an inch in diameter?

**8.** Three spheres of lead whose radii are 6 inches, 8 inches, and 10 inches respectively are melted and made into one. What is the radius of the resulting sphere?

**9.** On board a ship at sea the distance of the horizon varies directly as the square root of one's height above the water. If, at a height of 20 feet, the horizon is 5.5 miles distant, what is its distance as seen from a lighthouse 80 feet above sea-level?

**10.** The horse-power that a shaft can safely transmit varies jointly as its speed in revolutions per minute and the cube of its diameter. A 3-inch steel shaft making 100 revolutions per minute can transmit 85 horse-power. How many horse-power can a 4-inch shaft transmit at a speed of 150 revolutions per minute?

### II. ILLUSTRATIONS OF CASE (2)

**11.** Knowing that the force of gravitation due to the earth varies inversely as the square of the distance from the earth's center (*Newton's Law of Gravitation*), find how far above the earth's surface a body must be taken in order to lose half its weight.

SOLUTION. Letting W represent the weight of a given body at the distance $d$ from the earth's center, the law stated above, when expressed as an equation, becomes

$$(1) \qquad W=\frac{k}{d^2}. \quad (k=\text{some constant})$$

Now let $W_1$ represent the weight of the body when on the surface. Remembering that the earth's radius is 4000 miles (approximately), equation (1) gives

$$(2) \qquad W_1=\frac{k}{4000^2}.$$

Next, let $x$ represent the desired distance, namely the distance above the surface at which the same body loses half its weight. At this distance its weight will consequently be $\frac{1}{2}W_1$, while its distance from the earth's *center* is now $4000+x$. So (1) gives

$$(3) \qquad \frac{W_1}{2}=\frac{k}{(4000+x)^2}.$$

Dividing equation (3) by equation (2), noting the cancelation of $W_1$ on the left and of the (unknown) $k$ on the right, we obtain

$$\frac{1}{2}=\frac{4000^2}{(4000+x)^2}.$$

It remains only to solve this equation for $x$.

Clearing of fractions, $(4000+x)^2=2\cdot 4000^2=4000^2\cdot 2$.

Extracting the square root of both members, $4000+x=4000\sqrt{2}$.

Solving, $x=4000\sqrt{2}-4000=4000(\sqrt{2}-1)$ miles. *Ans.*

To find the approximate value of this answer, we have (see tables)

$$\sqrt{2}=1.41421$$

so that $x=4000(1.41421-1)=4000\times .41421=1656.84$ miles. *Ans.*

**12.** Show that the earth's attraction at a point on the surface is over 5000 times as strong as at the distance of the moon; that is, at the (approximate) distance of 280,000 miles.

[Hint. Call $W_1$ the weight of a given body on the surface, and let $W_2$ represent the weight of the same body at the distance of the moon from the earth's center. Then use the law expressed in (1) of the solution of Ex. 11.]

**13.** A book is being held at a distance of 2 feet from an incandescent lamp. How much nearer must it be brought in order that the illumination on the page shall be doubled? (See Ex. 11 (*c*), p. 91.)

**14.** If two like coins (such as quarter dollars) were melted and made into a single coin of the same thickness as the original, show that its diameter would be $\sqrt{2}$ times as great.

[Hint. Call $D$ the diameter of the given coins and $A$ the area of each. Note that the area of the new coin will then be $2A$. Use the result stated in Hint to Ex. 3, p. 93.]

**15.** Find the result in Ex. 14 when four equal-sized coins are used.

**16.** Show that a falling body will pass over the second 3 feet of its descent in about .4 of the time it takes it to pass over the first 3 feet. (See Ex. 10, p. 90.)

**17.** The time required for a pendulum to make a complete oscillation (swing forward and back) varies directly as the square root of its length. By how much must a 2-foot pendulum be shortened in order that its time of complete oscillation may be halved?

**18.** If the diameter of a sphere be increased by 10%, by what per cent will the volume be increased?

**19.** Show that if a city is receiving its water supply by means of a main from a reservoir, the supply can be increased 25% by increasing the diameter of the main by about 12%.

**20.** It is desired to build a ship similar in shape to one already in use but having a 40% greater cargo space (or hold). By what per cent must the beam (width of the ship) be increased?

[Hint. See the Hint to Ex. 6, p. 93.]

**50. Variation Geometrically Considered.** If a variable $y$ varies directly as another variable $x$, we know (§ 48) that this is equivalent to having the equation $y=kx$, where $k$ is some constant. If the value of $k$ is 1, this equation takes the definite form $y=x$, and we may now draw its graph, the result being a certain straight line. If, on the other hand, $k=2$, we have $y=2x$, and this again is an equation whose graph may be drawn, leading to a straight line, but a different one. In general, whatever the value of $k$, the corresponding equation has a straight-line graph. The fact that in all cases the graph is a *straight line* characterizes this type of variation; that is, characterizes the type in which one variable varies *directly* as another. Figure 36 shows the lines corresponding to several different values of $k$.

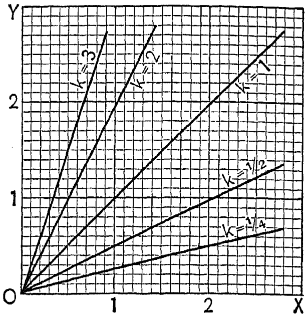

FIG. 36

In case a variable $y$ varies inversely as another variable $x$, we know (§ 48) that there exists an equation of the form $y=k/x$, where $k$ is some constant. If we let $k=1$, this becomes $y=1/x$. By letting $x$ take a series of values and determining the corresponding values of $y$ from this equation (thus forming a table as in § 25) we obtain the graph. Similarly, corresponding to the value $k=2$ we have $y=2/x$, and this equation has a definite graph which is different from the one just mentioned. In general, whatever the value of $k$,

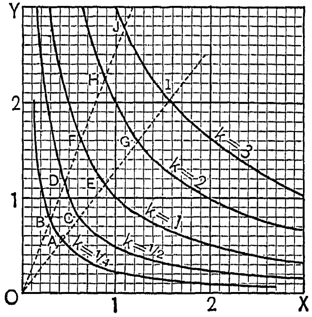

FIG. 37

corresponding equation has a graph, but it is now to be d that these graphs are *not* straight lines; they are *rbolas.* (See Ex. 2, § 28.) Figure 37 shows the curves esponding to several different values of $k$.

OTE. Though these curves differ in form, they have the following re in common: Through the origin draw any two straight lines ed in figure). Then the intercepted arcs $AB$, $CD$, $EF$, $GH$, etc., milar; that is, the smallest arc when simply *magnified* by the proper nt produces one of the others.

### EXERCISES

raw diagrams to represent the geometric meaning of each of the ving statements.

. $y$ varies directly as the square of $x$.

. $y$ varies inversely as the square of $x$.

. $y$ varies as the cube of $x$.

. $y$ varies directly as $x$, and $y=6$ when $x=2$.

HINT. The diagram here consists of a single line.]

. $y$ varies inversely as $x$, and $y=6$ when $x=2$.

. The cost of $n$ pounds of butter at 40c per pound is $C=40n$.

. The amount of the extension, $e$, of a stretched string is proporl to the tension, $t$, and $e=2$ in. when $t=10$ lb. (See Ex. 11 ($c$), .)

. The pressure, $p$, of a gas on the walls of a retaining vessel varies rsely as the volume, $v$; and $p=40$ lb. per square foot when $v=10$ t.

. The length, $L$, of any object in centimeters is proportional to ength, $l$, expressed in inches; and $L=2.54$ cm. when $l=1$ in.

# CHAPTER VII

## LOGARITHMS

### I. GENERAL CONSIDERATIONS†

**51. Definition of Logarithms.** If we ask what power of 10 must be used to give a result of 100, the answer is 2 because $10^2=100$. Another common way of stating this is to say that "the *logarithm* of 100 is 2." In the same way, the power of 10 needed to give 1000 is 3 because $10^3=1000$, and this is briefly stated by saying that "the *logarithm* of 1000 is 3." Similarly, the power of 10 that gives .1 is $-1$ because $10^{-1}=\frac{1}{10}$, or .1 by $(B)$, § 8, and this is equivalent to saying that "the *logarithm* of .1 is $-1$." Likewise, the logarithm of .01 is $-2$.

From these illustrations we readily see what is meant by the logarithm of a number. It may be defined as follows:

*The* ***logarithm*** *of a number is the power of* 10 *required to give that number.*

NOTE. A more general definition will be given in § 67, but this is the one commonly used in practice.

We write $\log 100=2$ to indicate that the logarithm of 100 is 2. Similarly, $\log 1000=3$, $\log .1=-1$, $\log .01=-2$, etc.

### EXERCISES

**1.** What is the *meaning* of log 10000? What is its *value*?

**2.** What is the value of log .001? Why?

**3.** What is the value of log .00001? Why?

**4.** What is the value of log 10?

**5.** What is the value of log 1? (See VIII, § 8.)

**6.** As a number increases from 100 to 1000, how does its logarithm change?

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†Parts I and II give definitions and essential theorems which should be well understood before Part III, which describes the important applications, is taken up.

**7.** As a number decreases from .1 to .01 how does its logarithm change? Answer the same as the number goes from .01 to .001; from 1 to 10; from 1 to 1000.

**8.** Explain why the following are true statements:

(*a*) $\log 100000 = 5$. (*b*) $\log .0001 = -4$.

(*c*) $\log \sqrt{10} = \frac{1}{2}$.

[HINT. Remember $\sqrt{10} = 10^{\frac{1}{2}}$.]

(*d*) $\log \sqrt[3]{10} = \frac{1}{3}$.

(*e*) $\log \sqrt[3]{100} = \frac{2}{3}$.

[HINT. Remember $\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{\frac{2}{3}}$. (§ 8.).]

(*f*) $\log \sqrt{.1} = -\frac{1}{2}$.

**52. Logarithm of Any Number.** Suppose we ask what the value is of log 236. What we are asking for (see definition in § 51) is that value which, when used as an exponent to 10, will give 236; that is, we wish the value of $x$ which will satisfy the equation $10^x = 236$. This question resembles those in § 51, but is different because we cannot immediately arrive at the desired value of $x$ *by mere inspection.* All we can say here at the beginning is that $x$ must lie *somewhere* between 2 and 3, because $10^2 = 100$ and $10^3 = 1000$, and 236 lies between these two numbers. In order to find $x$ to a finer degree of accuracy, it is now natural to try for it such values as 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, and 2.9, all of which lie between 2 and 3. The result (which for brevity we shall here state without proof) is that when $x = 2.3$ the value of $10^x$ is slightly *less* than our given number, 236, while if we take $x = 2.4$ the value of $10^x$ is slightly *greater* than 236. Thus $x$ lies somewhere between 2.3 and 2.4. In other words, the value of log 236 *correct to the first decimal place* is 2.3.

It is now natural, if we wish to obtain $x$ to still greater accuracy, to try for it such values as 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, and 2.39, all of which lie between 2.3 and 2.4 The result (which again is here stated without proof) is that when $x = 2.37$ the value of $10^x$ is slightly *less* than our number 236, while if we take $x = 2.38$ the value of $10^x$ is slightly *greater* than 236. This means that the second figure of the decimal is 7, after which we may say that the value of log 236 *correct to two places of decimals* is 2.37

Proceeding further in the same manner, it can be shown that when $x = 2.372$ the value of $10^x$ is slightly less than 236, while for $x = 2.373$ the value of $10^x$ is slightly greater than 236. Thus the value of log 236 *correct to three places of decimals* is 2.372 Similarly, it can be shown that the number in the fourth decimal place is 9, and this is as far as it is necessary to carry out the process, since the result is then sufficiently accurate for all ordinary purposes. Hence $\log 236 = 2.3729$, correct to four places of decimals.

NOTE. It thus appears that logarithms do not in general come out *exact*, though they do so for such exceptional numbers as 100, 1000, 10,000, .1, .01, etc. They can be expressed only approximately, yet as accurately as one pleases by carrying out the decimal far enough. In this respect they resemble such numbers as $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[4]{3}$, etc.

Other examples of logarithms are given below. Note especially the decimal part of each, which is correct to four places.

$$\log 283 = 2.4518 \qquad \log 196 = 2.2923 \qquad \log 17 = 1.2304$$
$$\log 6 = 0.7782 \qquad \log 3.410 = 0.5328 \qquad \log 5.75 = 0.7597$$

**53. Characteristic. Mantissa.** We have seen that the logarithm of a number consists (in general) of an integral part and a decimal part. These two parts of every logarithm are given special names as follows:

*The integral part of a logarithm is called the* ***characteristic*** *of the logarithm.*

*The decimal part of a logarithm is called the* ***mantissa*** *of the logarithm.*

Thus, since $\log 236 = 2.3729$, the characteristic of log 236 is 2, while its mantissa is .3729

Similarly, the characteristic of log 6 is 0, while its mantissa is .7782

## EXERCISES

**1.** What is the characteristic of log 100? What the mantissa? Answer the same questions for log 1000, log 10, and log 1.

**2.** What is the characteristic of log 185?

[HINT. Note that 185 lies between $10^2$ and $10^3$.]

**3.** What is the characteristic of log 310? of log 1287? of log 85? of log 21? of log 4? of log 12? of log 13987?

**4.** For what kind of number can one tell *by inspection* both the characteristic and the mantissa of its logarithm? (See § 51.)

**54. Further Study of Characteristic and Mantissa.** We have seen (§ 53) that log 236 = 2.3729, which is the same as saying that

$$10^{2.3729} = 236. \tag{1}$$

Let us now multiply both members of (1) by 10. The left side becomes $10^{2.3729+1}$ or $10^{3.3729}$ (§ 8, Formula I) while the right side becomes 2360. That is, we have $10^{3.3729} = 2360$, which is the same as saying that log 2360 = 3.3729

If, instead of multiplying both sides of (1) by 10, we *divide* both by 10, we obtain in like manner $10^{2.3729-1} = 23.6$ (§ 8, Formula V). That is, we have $10^{1.3729} = 23.6$, which is the same as saying that log 23.6 = 1.3729

Finally, if we divide both sides of (1) by $10^2$, or 100, we obtain $10^{2.3729-2} = 2.36$ That is, we have $10^{0.3729} = 2.36$ which is the same as saying that log 2.36 = 0.3729

What we now wish to do is to *compare* the results which we have just been obtaining, and for this purpose they are arranged side by side in a column below.

$$\begin{cases} \log 2360 = 3.3729 \\ \log 236 = 2.3729 \\ \log 23.6 = 1.3729 \\ \log 2.36 = 0.3729 \end{cases} \tag{2}$$

Note that the mantissas here appearing on the right are all the same, namely .3729, while the numbers appearing on the left (that is, 2360, 236, 23.6, and 2.36) are alike except for the position of the decimal point; that is, they contain the same significant figures. This illustrates the following important rule.

RULE I. *If two or more numbers have the same significant figures (that is, differ only in the location of the decimal point), their logarithms will have the* **same** *mantissas; that is, their logarithms can differ only in their characteristics.*

Thus, log 243, log 2430, log 24.3, log 2.43, log .243, and log .0243 all have the *same* mantissas. It is only their characteristics that can be different.

### EXERCISE

Apply Rule I, § 54, to tell which of the following logarithms have the same mantissas.

log .167
log .081
log 8100
log 7.2
log 16.7
log 720
log 81
log 1670
log .0072
log 16700

## II. TO DETERMINE THE LOGARITHM OF ANY NUMBER

**55. Purpose of This Part.** When we wish to determine the value of a logarithm, as for example, to find log 236, we can work out the characteristic and mantissa as explained in § 52, but this requires considerable time. What we do *in practice* is to use certain simple rules for determining the characteristic, and we determine the mantissa directly from certain tables which have been carefully prepared for the purpose. We shall now state these rules (§§ 56–58) and explain the tables and how to use them (§§ 59–61).

**56. Characteristics for Numbers Greater Than 1.** If we look again at the results in (2) of § 54, we see that the characteristic of log 2360 is 3. Thus the characteristic is 1 *less* than the number of figures to the *left* of the decimal point.

NOTE. 2360 is the same as 2360., so that there are *four* figures here to the left of the decimal point.

Again, we see from (2) of § 54 that the characteristic of log 236 is 2 and this, as in the case already examined, is 1 less than the number of figures to the left of the decimal point.

**Note.** 236 is the same as 236., so there are *three* figures here to the left of the decimal point.

Similarly, since the characteristic of log 23.6 is 1 (see (2) of § 54) this again obeys the same law as just observed in the other two cases; that is, the characteristic is 1 less than the number of figures to the left of the decimal point.

Finally, since the characteristic of log 2.36 is 0, the same law is again present here.

The law which we have just observed can be shown in like manner to hold good for the characteristic of the logarithm of any number greater than 1; hence we may state the following general rule.

Rule II. *The characteristic of the logarithm of a number greater than* 1 *is one less than the number of figures to the left of the decimal point.*

Thus, the characteristic of log 385.9 is 2; that of log 8.679 is 0.

### EXERCISES

State, by Rule II, § 56, the characteristic of the logarithm of each of the following numbers.

**1.** 385.4
**2.** 461.
**3.** 7962.
**4.** 2.7
**5.** 75.54
**6.** 165,781
**7.** 18.831
**8.** 3.1568
**9.** 401.005
**10.** 2967.6
**11.** 85.
**12.** 2.46879

State how many figures precede the decimal point of a number it the characteristic of its logarithm is

**13.** 2.
**14.** 3.
**15.** 1.
**16.** 0.
**17.** 5.
**18.** 4.

**57. Characteristics for Positive Numbers Less Than 1.** We have seen (see (2) in § 54) that log $2.36 = 0.3729$, which is the same as saying that

$$10^{0.3729} = 2.36 \tag{1}$$

Let us now divide both members of this relation by 10. We thus obtain (§ 8, Formula V)

$$10^{0.3729-1} = .236 \quad \text{or } 10^{-1+0.3729} = .236,$$

which means (by § 51)

$$\log .236 = -1 + 0.3729$$

Observe that $-1+0.3729$ is really a negative quantity, being equal to $-(1-0.3729)$ which reduces to $-0.6271$ However, it is more convenient for our present purposes to keep the longer form $-1+0.3729$ Note that this *cannot* be written as $-1.3729$ because the latter is equal to $-1-0.3729$ instead of $-1+0.3729$

If, instead of dividing both members of (1) by 10, we divide both by $10^2$, or 100, we obtain

$$10^{0.3729-2} = .0236 \quad (\text{or } 10^{-2+0.3729} = .0236),$$

which means that

$$\log .0236 = -2 + 0.3729$$

Similarly, by dividing (1) by $10^3$, or 1000, we find that

$$\log .00236 = -3 + 0.3729$$

Finally, if we divide (1) by $10^4$, or 10000, we find that

$$\log .000236 = -4 + 0.3729$$

Let us now compare the four results just obtained. Beginning with the last result, we see that in the number .000236 there are *three* zeros immediately to the *right* of the decimal point; that is, between the decimal point and the first significant figure. Corresponding to this, the characteristic on the right is *minus four.* Hence the characteristic is negative and 1 more numerically than the number of zeros between the decimal point and the first significant figure.

Similarly, in the number .00236 there are *two* zeros between the decimal point and the first significant figure, and corresponding to this there is a characteristic on the right of *minus three.* Hence, as before, the characteristic here is negative and numerically 1 more than the number of zeros between the decimal point and the first significant figure. This statement, which is true in all cases mentioned above, can be proved for the characteristic of the logarithm of any positive number less than 1. Hence we have the following rule.

RULE III. *The characteristic of the logarithm of a (positive) number less than* 1, *is negative, and is numerically* 1 *greater than the number of zeros between the decimal point and the first significant figure.*

Thus, the characteristic of log .0076 is $-3$; that of log .28 is $-1$.

**NOTE.** The logarithm of a negative number is an imaginary quantity (as shown in higher mathematics), and hence we shall consider here the logarithms of positive numbers only.

### 58. Usual Method of Writing a Negative Characteristic.

In § 57 we saw that

$$\log .236 = -1 + 0.3729$$

If we add 10 to this quantity and at the same time subtract 10 from it we do not change its value, but we give it the new form

$$9 + 0.3729 - 10,$$

which is the same as $9.3729 - 10$. That is, we may write

$$\log .236 = 9.3729 - 10.$$

This is the form used in practice.

Likewise, instead of writing $\log .0236 = -2 + 0.3729$ (see § 57) we write in practice

$$\log .0236 = 8.3729 - 10,$$

and similarly we write

$$\log .00236 = 7.3729 - 10.$$

Thus, the usual method of expressing the characteristic whose value is $-1$ is to write $9-10$ for it; if it is $-2$, we write $8-10$ for it; if it is $-3$, we write $7-10$ for it, etc.

For example, log .0076 has the characteristic 7–10.

### EXERCISES

State, by Rule III, § 57, the value of the characteristic of the logarithm of each of the following; state how it would be written if expressed in the usual form described in § 58.

**1.** .06 $-2$, or $8-10$. *Ans.*
**2.** .0071
**3.** .81
**4.** .00053
**5.** .835
**6.** .0835
**7.** .4578
**8.** .00875
**9.** .15681
**10.** .00005

How many zeros lie between the decimal point and the first significant figure of a number when the characteristic of its logarithm is

**11.** $-3$
**12.** $9-10$
**13.** $-5$
**14.** $8-10$
**15.** $7-10$
**16.** $6-10$

**59. Determination of Mantissas. Use of Tables.** Suppose we wish to determine completely the value of log 187. By Rule II, § 56, we know that the characteristic is 2. To find the mantissa, we turn to the tables (p. 108) and look in the column headed **N** for the first two figures of the given number, that is, for 18. The desired mantissa is then to be found on the horizontal line with these two figures and in the column headed by the third figure of the given number; that is, in the column headed by 7. Thus in the present case the mantissa is found to be .2718

NOTE. For brevity, the decimal point preceding each mantissa is omitted from the tables. It must be supplied as soon as the mantissa is used.

The complete value (correct to four decimal places) of log 187 is therefore 2.2718

Again, suppose we wish to determine log 27.6. The characteristic (by § 56) is 1. The mantissa, by Rule I, § 54, is the same as that of log 276; and the latter, as given in the tables, is .4409 Therefore, log 27.6 = 1.4409 *Ans.*

As a last example, suppose we wish to determine log .0173 The characteristic (by § 57) is −2, or 8−10. The mantissa, by the rule in § 54, is the same as that of log 173 and the latter, as obtained from the tables, is .2380 Therefore, log .0173 = 8.2380 − 10. *Ans.*

These examples illustrate how the tables together with Rules II and III, §§ 56, 57, enable us to determine completely the logarithm of any number provided it contains no more than three significant figures. We may now summarize our results in the following rule.

RULE IV. *To find the logarithm of a number of three significant figures:*

1. *Look in the column headed* **N** *for the first two figures of the given number. The mantissa will then be found on the horizontal line opposite these two figures and in the column headed by the third figure of the given number.*

2. *Prefix the characteristic according to Rules II and III,* §§ 56, 57.

### EXERCISES

Determine the logarithm of each of the following numbers, expressing all negative characteristics as explained in § 58.

**1.** 561 **2.** 217 **3.** 280 **4.** 800
**5.** 72.5 [HINT TO EX. 5. Note how log 27.6 was obtained in § 59.]
**6.** 7.25 **7.** 93. **8.** 9. **9.** .0136
**10.** .936 **11.** .0036 [HINT. Write as .00360]
**12.** 7550.
**13.** .071
**14.** .7
**15.** .35
**16.** 55.7
**17.** 25,300
**18.** .000831
**19.** $\frac{1}{2}$.
**20.** $\frac{2}{3}$.

| **N** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **10** | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 |
| **11** | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 |
| **12** | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 |
| **13** | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 |
| **14** | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 |
| **15** | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 |
| **16** | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 |
| **17** | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 |
| **18** | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 |
| **19** | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 |
| **20** | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 |
| **21** | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 |
| **22** | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 |
| **23** | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 |
| **24** | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 |
| **25** | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 |
| **26** | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 |
| **27** | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 |
| **28** | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 |
| **29** | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 |
| **30** | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 |
| **31** | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 |
| **32** | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 |
| **33** | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 |
| **34** | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 |
| **35** | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 |
| **36** | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 |
| **37** | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 |
| **38** | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 |
| **39** | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 |
| **40** | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 |
| **41** | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 |
| **42** | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 |
| **43** | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 |
| **44** | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 |
| **45** | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 |
| **46** | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 |
| **47** | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 |
| **48** | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 |
| **49** | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 |
| **50** | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 |
| **51** | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 |
| **52** | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 |
| **53** | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 |
| **54** | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 |

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **55** | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 |
| **56** | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 |
| **57** | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 |
| **58** | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 |
| **59** | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 |
| **60** | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 |
| **61** | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 |
| **62** | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 |
| **63** | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 |
| **64** | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 |
| **65** | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 |
| **66** | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 |
| **67** | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 |
| **68** | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 |
| **69** | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 |
| **70** | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 |
| **71** | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 |
| **72** | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 |
| **73** | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 |
| **74** | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 |
| **75** | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 |
| **76** | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 |
| **77** | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 |
| **78** | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 |
| **79** | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 |
| **80** | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 |
| **81** | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 |
| **82** | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 |
| **83** | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 |
| **84** | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 |
| **85** | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 |
| **86** | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 |
| **87** | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 |
| **88** | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 |
| **89** | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 |
| **90** | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 |
| **91** | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 |
| **92** | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 |
| **93** | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 |
| **94** | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 |
| **95** | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 |
| **96** | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 |
| **97** | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 |
| **98** | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 |
| **99** | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 |

**60. To Find the Logarithm of a Number of More Than Three Significant Figures.** Suppose we wish to determine log 286.7 Here we have *four* significant figures, while our tables tell us the mantissas of numbers having three (or less) significant figures (as in § 59 and in the preceding exercises). In such cases we proceed as follows.

From the tables on pp. 108–109 we have

$$\left.\begin{array}{r}\log 286 = 2.4564\\ \log 286.7 = ?\phantom{0000}\\ \log 287 = 2.4579\end{array}\right\}\text{Difference} = 2.4579 - 2.4564 = .0015$$

Since 286.7 lies between 286 and 287, its logarithm must lie between their logarithms. Now, an increase of one unit in the number (in going from 286 to 287) produces an increase of .0015 in the mantissa. It is therefore assumed that an increase of .7 in the number (in going from 286 to 286.7) produces an increase of

$$.7 \text{ of } .0015, \text{or } .00105,$$

in the mantissa.

Therefore,

$$\log 286.7 = 2.4564 + .7 \text{ of } .0015 = 2.4564 + .00105 = 2.45745,$$

so that

$$\log 286.7 = 2.4574 \text{ (approximately)}. \quad \textit{Ans.}$$

In practice the answer is quickly obtained as follows:

The difference between any mantissa and the next higher one in the table (neglecting the decimal point) is called the *tabular difference.* The tabular difference in this example is

$$4579 - 4564, \text{ or } 15.$$

Taking .7 of this, we obtain 10.5, which (keeping only the first two figures) we call 10, and adding this to 4564 we find 4574. This, therefore, is the required mantissa of log 286.7, so that

$$\log 286.7 = 2.4574 \text{ (approximately)}.$$

Similarly, in finding log 286.75 the tabular difference (as before) is 15. Taking .75 of 15 gives 11.25, which (keeping only two figures) has the approximate value 11.

Hence the mantissa of log 286.75 is $4564+11=4575$. Therefore log $286.75=2.4575$ *Ans.*

Below are two examples further illustrating how the above processes are quickly carried out in practice. The student should form the habit of writing the work in this form.

EXAMPLE 1. Determine the value of log 48.731

SOLUTION.
$$\left.\begin{array}{l}\text{Mantissa of log } 487=6875\\ \text{Mantissa of log } 488=6884\end{array}\right\}\text{Tabular difference}=9$$

$$.31\times 9=2.79=3 \text{ (approximately).}$$

Hence

$$\text{mantissa of log } 48.731=6875+3=6878.$$

Therefore

$$\log 48.731=1.6878. \quad \textit{Ans.}$$

EXAMPLE 2. Determine the value of log .013403

SOLUTION.
$$\left.\begin{array}{l}\text{Mantissa of } 134=1271\\ \text{Mantissa of } 135=1303\end{array}\right\}\text{Tabular difference}=32.$$

$$.03\times 32=.96=1 \text{ (approximately).}$$

Hence

$$\text{mantissa of log } .013403=1271+1=1272.$$

Therefore

$$\log .013403=-2+.1272=8.1272-10. \quad \textit{Ans.}$$

**NOTE.** The process which we have employed for determining a mantissa when it does not actually occur in the tables is called ***interpolation***. When examined carefully, it will be seen that the process is based upon the assumption that if a number is increased by any fractional amount of itself, the logarithm of the number will likewise be increased by the *same* fractional amount of itself. Thus, in finding the mantissa of log 286.7 at the middle of p. 110, we assumed that the increase of .7 in going from 286 to 286.7 would be accompanied by like increase of .7 in the logarithm. Such an assumption, though not *exactly* correct, is very nearly so in most cases and is therefore sufficiently accurate for all ordinary purposes.

Tables of logarithms much more extensive than those on pages 108, 109 have been prepared and are commonly used. See, for example, *The Macmillan Tables.* By means of these, any desired mantissa may usually be obtained as accurately as is necessary directly, that is, without interpolation.

### EXERCISES

Obtain the logarithm of each of the following numbers.

**1.** 578.3

**2.** 332.2

**3.** 675.3

**4.** 481.6

**5.** 956.7

**6.** 22.17

**7.** 8.467

**8.** 3.706

**9.** 2.408

**10.** 2.767

**11.** .3456

**12.** .07235

**13.** 745.23

**14.** 132.36

**15.** 51.745

**16.** 430.07

**17.** 5.2178

**18.** 4.2316

**19.** 1.6086

**20.** .14653

**21.** .074568

**22.** .00738

**61. To Find the Number Corresponding to a Given Logarithm.** Thus far we have considered how to determine the logarithm of a given number, but frequently the problem is reversed, that is, it is the logarithm that is given and we wish to find the number having that logarithm. The method of doing this is the reverse of the method of §§ 59, 60, and is illustrated in the following examples.

Example 1. Find the number whose logarithm is 1.9547

Solution. Locate 9547 among the mantissas in the table. Having done so, we find in the column N on the line with 9547 the figures 90. These form the first two figures of the desired number.

At the head of the column containing 9547 is 1, which is therefore the third figure of the desired number.

Hence the number sought is made up of the digits 901.

The given characteristic being 1, the number just found must be pointed off so as to have *two* figures to the left of its decimal point (Rule II, § 56). Therefore the number is 90.1 *Ans.*

Example 2. Find the number whose logarithm is 0.6341

Solution. As in Example 1, we look among the mantissas of the table to find 6341. In this case we do not find *exactly* this mantissa, but we see that the next less mantissa appearing is 6335, while the one next greater is 6345.

The numbers corresponding to these last two mantissas are seen to be 430 and 431 respectively. Whence, if $x$ represents the number sought, we have

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{Mantissa of log } 430 = 6335\\ \text{Mantissa of log } x \quad\; = 6341\end{array}\right\}\text{Diff.} = 6\\ \text{Mantissa of log } 431 = 6345\end{array}\right\}\text{Tabular difference} = 10.$$

Since an increase of 10 in the mantissa produces an increase of 1 in the number, we assume that an increase of 6 in the mantissa will produce an increase of $\frac{6}{10}$, or .6, in the number.

Hence the number sought has the digits 4306.

Since the given characteristic is 0, it is evident that the number must be 4.306 (§ 56). *Ans.*

**Note 1.** The student will observe that in Example 1 the given mantissa actually occurs in the tables, while in Example 2 it does not, thus making it necessary in this last case to interpolate. (See the Note in § 60.)

**Note 2.** The number whose logarithm is a given quantity is called the ***antilogarithm*** of that quantity. Thus 100 is the antilogarithm of 2; 1000 is the antilogarithm of 3, etc.

## EXERCISES

Find the numbers whose logarithms are given below.

**1.** 2.6656
**2.** 1.8351
**3.** 0.2742
**4.** 2.5855
**5.** $9.6830-10$
**6.** $8.8028-10$
**7.** $7.6425-10$
**8.** $6.8842-10$
**9.** 1.2517
**10.** 2.8583
**11.** 3.7430
**12.** 0.5240
**13.** 0.6970
**14.** $9.7400-10$
**15.** $8.3090-10$
**16.** $7.5308-10$
**17.** $9.0046-10$
**18.** $8.0012-10$
**19.** $3.4968-10$
**20.** $5.9654-10$

## III. The Use of Logarithms in Computation

**62. To Find the Product of Several Numbers.** The processes of multiplication, division, raising to powers, and extraction of roots, as carried out in arithmetic, may be greatly shortened by the use of logarithms, as we shall now show.

Let us take any two numbers, for example 25 and 37, and determine their logarithms. We find that $\log 25 = 1.3979$ and $\log 37 = 1.5682$ This means (§ 136) that

$$25 = 10^{1.3979} \quad \text{and} \quad 37 = 10^{1.5682}$$

Multiplying, we thus have

$$25 \times 37 = 10^{1.3979 + 1.5682} \qquad (\text{§ 8, Formula I})$$

The last equality means (§ 51) that

$$\log (25 \times 37) = 1.3979 + 1.5682,$$

or

$$\log (25 \times 37) = \log 25 + \log 37.$$

Similarly, if we start with the *three* numbers 25, 37, and 18 we can show that

$$\log (25 \times 37 \times 18) = \log 25 + \log 37 + \log 18.$$

Thus we arrive at the following important rule.

Rule V. *The logarithm of a product is equal to the* ***sum*** *of the logarithms of its factors.*

Thus $\log (13 \times .0156 \times 99.8) = \log 13 + \log .0156 + \log 99.8$

The way in which this rule is used to find the value of the product of several numbers is shown below.

Example 1. To find the value of $13 \times .0156 \times 99.8$

Solution.
$$\begin{aligned} \log 13 &= 1.1139 \\ \log .0156 &= 8.1931 - 10 \\ \log 99.8 &= 1.9991 \\ \hline \text{Adding,} \quad & 11.3061 - 10, \text{ or } 1.3061 \end{aligned}$$

Hence, by Rule V, the logarithm of the desired product is 1.3061 It follows that the product itself is the number whose logarithm is 1.3061 When we look up this number (as in § 61) we find it to be 20.23 Hence $13 \times .0156 \times 99.8 = 20.23$ (approximately). *Ans.*

EXAMPLE 2. To find the value of

$$8.45 \times .678 \times .0015 \times 956 \times .111$$

SOLUTION.

$$\begin{aligned} \log 8.45 &= 0.9269 \\ \log .678 &= 9.8312-10 \\ \log .0015 &= 7.1761-10 \\ \log 956 &= 2.9805 \\ \log .111 &= \underline{9.0453-10} \end{aligned}$$

Adding, $29.9600-30=9.9600-10.$

Hence, by Rule V, the logarithm of the desired product is $9.9600-10$.

Therefore the product itself is found (as in § 61) to be .912 (approximately). *Ans.*

These examples illustrate the following rule.

RULE VI. *To multiply several numbers:*

1. *Add the logarithms of the several factors.*
2. *The sum thus obtained is the logarithm of the product.*
3. *The product itself can then be determined as in* § 61.

### EXERCISES

Find, by Rule V, § 62, the value of each of the following logarithms.

**1.** $\log (38.2 \times 6.31)$.

**2.** $\log (6 \times 4.21 \times .0015)$.

**3.** $\log (167 \times 7.31 \times .00456)$.

**4.** $\log (3.81 \times .00175 \times 1.87)$.

Find, by Rule VI, § 62, the value of the following products. Check your answer in Ex. 5 by multiplying out the long way as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method. Compare also the time required for the two methods.

**5.** $56.8 \times 3.47 \times .735$

**6.** $.975 \times 42.8 \times 3.72$

**7.** $896 \times 40.8 \times 3.75 \times .00489$

**8.** $34.56 \times 18.16 \times .0157$

[HINT. See § 60.]

**9.** $576.8 \times 43.25 \times 3.576 \times .0576$

**10.** $60.573 \times 8.087 \times .008915 \times 1.2387$

**11.** $23 \times 23 \times 23 \times 23 \times 23 \times 23 \times 23$, (or $23^7$).

**12.** $1.2 \times 2.3 \times 3.4 \times 4.5 \times 5.6 \times 6.7 \times 7.8$

**13.** $.31 \times 5.198 \times 6.831 \times 2.584 \times .00312 \times .07568$

**14.** Since $25 \times 15 = 375$ we know by Rule V, § 62, that the logarithm of 25 added to the logarithm of 15 is equal to the logarithm of 375. Show that the values given in the tables for log 25, log 15, and log 375 confirm this result. Invent and try out several other similar problems.

**63. To Find the Quotient of Two Numbers.** Let us take any two numbers, for example 41 and 29, and write their logarithms. We find

$$\log 41 = 1.6128$$
$$\log 29 = 1.4624$$

These mean that

$$41 = 10^{1.6128}$$

and

$$29 = 10^{1.4624}$$

Whence, dividing the first of these equalities by the second, we obtain

$$41 \div 29 = \frac{10^{1.6128}}{10^{1.4624}} = 10^{1.6128-1.4624} \qquad (\S\ 8,\ \text{Formula V})$$

The last equality means that

$$\log (41 \div 29) = 1.6128 - 1.4624 = \log 41 - \log 29.$$

This result illustrates the following general rule.

RULE VII. *The logarithm of a quotient is equal to the logarithm of the dividend* **minus** *the logarithm of the divisor.*

Thus $\log (467.3 \div .00149) = \log 467.3 - \log .00149$

The way in which this rule is used is shown below.

EXAMPLE 1. To find the value of $236 \div 4.15$

SOLUTION.

$$\begin{aligned} \log 236 &= 2.3729 \\ \log 4.15 &= \underline{0.6180} \end{aligned}$$

Subtracting $\quad 1.7549$

Hence the logarithm of the desired quotient is 1.7549 (Rule VII.)

The number whose logarithm is 1.7549 is found (as in § 61) to be 56.875

Therefore $\quad 236 \div 4.15 = 56.875$ (approximately). *Ans.*

EXAMPLE 2. To find the value of $1.46 \div .00578$

SOLUTION.

$$\begin{aligned} \log 1.46 = 0.1644 &= 10.1644 - 10 \quad (\text{See Note p. 117.}) \\ \log .00578 \quad &= \underline{\ 7.7619 - 10} \end{aligned}$$

Subtracting, $\quad 2.4025$

The number whose logarithm is 2.4025 is found to be 252.64

Therefore $\quad 1.46 \div .00578 = 252.64$ (approximately). *Ans.*

Thus we have the following rule.

RULE VIII. *To find the quotient of two numbers:*

1. *Subtract the logarithm of the divisor from the logarithm of the dividend.*

2. *The difference thus obtained is the logarithm of the quotient.*

3. *The quotient itself can then be determined as in* § 61.

**NOTE.** To subtract a negative logarithm from a positive one, or to subtract a greater logarithm from a less, increase the characteristic of the minuend by 10, writing $-10$ after the mantissa to compensate. Thus, in Example 2, we wished to subtract the negative logarithm $7.7619-10$ from the positive one 0.1644 Therefore 0.1644 was written in the form $10.1644-10$, after which the subtraction was easily performed.

### EXERCISES

Find, by Rule VII, § 63, the value of each of the following logarithms.

**1.** $\log(17 \div 8)$.

**2.** $\log(218 \div 7.15)$.

**3.** $\log(37.5 \div .0018)$.

**4.** $\log(8.69 \div 113)$.

Find, by Rule VIII, § 63, the value of each of the following quotients. Check your answer in Ex. 5 by dividing out the long way as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method.

**5.** $246 \div 15.7$

**6.** $34.7 \div 5.34$

**7.** $389.7 \div 4.353$

[HINT. See § 60.]

**8.** $45.67 \div 38.01$

**9.** $3.25 \div .00876$

[HINT. See Note in § 63.]

**10.** $49.6 \div 87.3$

**11.** $\dfrac{40.3 \times 6.35}{3.72}$

[HINT. Find the logarithm of the numerator by Rule V, § 62.]

**12.** $\dfrac{.0036 \times 2.36}{.0084}$

**13.** $\dfrac{24.3 \times .695 \times .0831}{8.40 \times .216}$

**14.** Since $27 \div 9 = 3$ we know, by Rule VII, § 62, that the logarithm of 9 subtracted from the logarithm of 27 is equal to the logarithm of 3. Show that the values given in the tables for log 9, log 27, and log 3 confirm this result. Invent and try out several other similar problems for yourself.

**64. To Raise a Number to a Power.** Let us take any number, for example 25, and raise it to any power, say the fourth. We then have $25^4$, which means $25 \times 25 \times 25 \times 25$.

Hence, by Rule V, § 62, we have

$$\log 25^4 = \log 25 + \log 25 + \log 25 + \log 25, \text{ or } \log 25^4 = 4 \log 25.$$

This illustrates the following rule.

RULE IX. *The logarithm of any power of a number is equal to the logarithm of the number* ***multiplied*** ***by*** *the exponent indicating the power.*

Thus $\log 3.17^{10} = 10 \log 3.17$; similarly, $\log .00174^6 = 6 \log .00174$

The way in which this principle is used to raise a number to a power is shown below.

EXAMPLE 1. To find the value of $2.37^4$

SOLUTION. $\log 2.37 = 0.3747$

$$\underline{\phantom{0.374}4}$$

Multiplying, $1.4988$

Hence

$$\log 2.37^4 = 1.4988 \quad \text{(Rule IX)}$$

The number whose logarithm is 1.4988 is found to be 31.535

Therefore

$$2.37^4 = 31.535 \text{ (approximately)}. \quad \textit{Ans.}$$

EXAMPLE 2. To find the value of $.856^5$

SOLUTION. $\log .856 = 9.9325 - 10$

$$\underline{\phantom{9.9325-1}5}$$

Multiplying, $49.6625 - 50 = 9.6625 - 10$

The number whose logarithm is $9.6625 - 10$ is .4597

Therefore

$$.856^5 = .4597 \text{ (approximately)}. \quad \textit{Ans.}$$

Thus we have the following rule

RULE X. *To raise a number to a power:*

1. *Multiply the logarithm of the number by the exponent indicating the power.*
2. *The result thus obtained is the logarithm of the answer.*
3. *The answer itself can then be determined as in* § 61.

### EXERCISES

Find, by Rule IX, § 64, the value of each of the following logarithms.

**1.** $\log 16^5$ **2.** $\log 3.12^3$ **3.** $\log .0176^2$ **4.** $\log 36.64^4$

Find, by Rule X, § 64, the value of each of the following expressions.

**5.** $8.82^3$

Check your answer by raising 8.82 to the third power as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method.

**6.** $4.12^4$ **7.** $4.123^4$

**8.** $.175^5$ [Hint. See Ex. 2 in § 64.]

**9.** $81^3 \times .015^2$ [Hint. Combine the rules of §§ 62 and 64.]

**10.** $43 \times 8.9^2 \times .075^3$

**11.** $\dfrac{8.76 \times 53.9 \times 4.5^3}{2.3^2 \times 3.15 \times 5.14^3}$

[Hint. Use Rules VI, VIII, X.]

**12.** Since $9^3 = 729$ we know, by Rule IX, § 64, that three times the logarithm of 9 is equal to the logarithm of 729. Show that the values given in the tables for log 9 and log 729 confirm this result. Invent and try out several other similar problems for yourself.

**65. To Extract Any Root of a Number.** Let us take any number, for example 36, and consider any root of it, say the fifth; that is, let us consider $\sqrt[5]{36}$.

Supposing $x$ to be the value of the desired root, we have

$$x^5 = 36.$$

Now the logarithm of the first member of this equality is equal to $5 \log x$ by Rule IX.

Hence $5 \log x = \log 36$, or $\log x = \frac{1}{5} \log 36$.

This illustrates the following rule.

Rule XI. *The logarithm of the root of a number is equal to the logarithm of the radicand* ***divided by*** *the index of the root.*

Thus $\log \sqrt[4]{2.73} = \frac{1}{4} \log 2.73$; similarly, $\log \sqrt[7]{.01685} = \frac{1}{7} \log .01685$

The way in which this principle is used to extract the roots of numbers in arithmetic will now be shown.

Example 1. To find the value of $\sqrt[4]{85.2}$

Solution. $\log 85.2 = 1.9304$,

so that $\frac{1}{4}$ of $\log 85.2 = 0.4826$

Hence $\log \sqrt[4]{85.2} = 0.4826$ (Rule XI)

The number whose logarithm is 0.4826 is 3.038 (§ 61)

Therefore $\sqrt[4]{85.2} = 3.038$ (approximately). *Ans.*

Example 2. To find the value of $\sqrt[5]{.0875}$

Solution. $\log .0875 = 8.9420 - 10$,

so that $\frac{1}{5}$ of $\log .0875 = \frac{1}{5}(8.9420 - 10) = \frac{1}{5}(48.9420 - 50)$

$= 9.7884 - 10$. (See Note below.)

The number whose logarithm is $9.7884 - 10$ is .6143 (§ 61)

Therefore $\sqrt[5]{.0875} = .6143$ (approximately). *Ans.*

These examples illustrate the following rule.

Rule XII. *To find any root of any number.*

1. *Divide the logarithm of the number by the index of the root.*
2. *The quotient obtained is the logarithm of the desired root.*
3. *The root itself can then be determined as in* § 61.

**Note.** To divide a negative logarithm, write it in a form where the negative part of the characteristic may be divided exactly by the divisor giving $-10$ as quotient as in Example 2.

## EXERCISES

Find, by Rule XI, § 65, the value of each of the following logarithms.

**1.** $\log \sqrt[5]{16}$. **2.** $\log \sqrt[3]{3.12}$ **3.** $\log \sqrt[4]{.0175}$ **4.** $\log \sqrt[5]{38.56}$

Find, by Rule XII, § 65, the value of each of the following expressions. Check your answer in Ex. 5 by extracting the square root of 315 (correct to three decimal places) as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method.

**5.** $\sqrt{315}$

**6.** $\sqrt[3]{4.32}$

**7.** $\sqrt[3]{4.325}$

**8.** $\sqrt[5]{.0957}$

[Hint. See Example 2 in § 65.]

**9.** $\sqrt[4]{8.76 \times .0153}$

[Hint. Use Rules IX and XI.]

**10.** $\sqrt[3]{576} \times \sqrt[4]{8.76}$

**11.** $\sqrt{\dfrac{576 \times 9.13^2}{3.8 \times 5.32^3}}$.

## APPLIED PROBLEMS

Solve the following exercises by logarithms.

**1.** How many cubic feet of air are there in a schoolroom whose dimensions are 50.5 ft. by 25.3 ft. by 10.4 ft.?

**2.** How many gallons will a rectangular tank hold whose dimensions are 8 ft. 10 in. by 9 ft. 3 in. by 10 ft. 1 in.?

**3.** How much wheat will a cylindrical bin hold if the diameter of the base is 9 ft. 5 in. and the height is 40 ft. 4 in.?

**4.** How much would a sphere of solid cork weigh if its diameter was 4 ft. 3 in., it being known that the specific gravity of cork is .24?

[HINT. To say that the specific gravity of cork is .24 means that any volume of cork weighs .24 times as much as an *equal* volume of water. Water weighs 62.5 pounds per cubic foot.]

**5.** The diameter $d$ in inches of a wrought-iron shaft required to transmit $h$ horse-power at a speed of $n$ revolutions per minute is given by the formula $d=\sqrt[3]{\frac{65h}{n}}$. Find the diameter required when 135 horse-power is to be transmitted at a speed of 130 revolutions per minute.

**6.** A wire 135 feet long is suspended from two poles of equal height placed 130 feet apart. Compute the sag, using the formula of Ex. 20, page 42.

**7.** If the three sides of a triangle are of lengths $a$, $b$, $c$ respectively, and we place $s=\frac{1}{2}(a+b+c)$, then the area is expressed by the formula

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

Determine the area of the triangle whose sides are 3.15 inches, 4.87 inches and 2.68 inches.

**8.** The height $H$ of a mountain in feet is given by the formula

$$H=49{,}000\left(\frac{R-r}{R+r}\right)\left(1+\frac{T+t}{900}\right),$$

where $R$, $r$ are the observed heights of the barometer in inches at the foot and at the summit of the mountain, and where $T$, $t$ are the observed Fahrenheit temperatures at the foot and summit.

Find the height of a mountain if the height of the barometer at the foot is 29.6 inches and at the summit 25.35 inches, while the temperature at the foot is 67° and at the summit 32°.

**66. Solution of Exponential Equations.** The equation

(1) $$2^x = 32,$$

wherein the unknown number, $x$, appears in the exponent, is an example of an ***exponential equation.*** In the present instance, the equation may be solved immediately by inspection, $x$ being equal to 5, since $2^5 = 32$. But if, instead of (1), we start with the following equally simple exponential equation

(2) $$2^x = 48$$

the value of $x$ can be obtained only approximately, and its determination involves the use of logarithms in the manner shown below:

Solution. Taking the logarithm of each member in (2),

$$x \log 2 = \log 48. \quad \text{(Rule IX)}$$

Therefore

$$x = \frac{\log 48}{\log 2} = \frac{1.6812}{0.3010} = 5.58^{+} \ \textit{Ans.}$$

### EXERCISES

Solve each of the following exponential equations, using logarithms.

**1.** $4^x = 10$.

**2.** $2^x = 80$.

**3.** $31^x = 23$.

**4.** $.2^x = 3$.

**5.** $13^x = .281$

**6.** $3^{2x} - 20 \cdot 3^x + 99 = 0$.

[Hint. $3^{2x} - 20 \cdot 3^x + 99 = (3^x - 9)(3^x - 11)$.

**7.** $\begin{cases} 3^x = 2y, \\ 2^x = y. \end{cases}$

**8.** $\begin{cases} 2^{x+y} = 6, \\ 2^{x+1} = 3^y. \end{cases}$

## IV. General Logarithms

***67. Logarithms to Any Base.** In § 51 we defined the logarithm of a number as the power to which 10 must be raised to obtain that number. Thus, from such equalities as $10^2 = 100$, $10^3 = 1000$, etc., we had $\log 100 = 2$, $\log 1000 = 3$, etc. Strictly speaking, this defines the logarithm of a number *to the base 10*, or, as it is usually called, a ***common logarithm.***

We may and frequently do use some other base than 10. For example, since $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, etc., we can say that the logarithm of 9 *to the base 3* is 2, the logarithm of 27 *to the base 3* is 3, the logarithm of 81 *to the base 3* is 4, etc. The usual way of denoting this is to write $\log_3 9 = 2$, $\log_3 27 = 3$, $\log_3 81 = 4$, etc. The number being used as the base is placed to the right and just below the symbol log.

Similarly, we have $\log_2 16 = 4$, $\log_8 64 = 2$, $\log_5 125 = 3$, etc.

Thus we have the following general definition. *The logarithm of any number $x$ to a given base $a$ is the power of $a$ required to give $x$. It is written* $\log_a x$. Any positive number except 1 may be used as the base.

**Note.** When the base $a$ is taken equal to 10 (that is, in the usual case) we write simply $\log x$ instead of $\log_{10} x$

### *EXERCISES

State first the *meaning* and then the *value* of

**1.** $\log_2 4$.
**2.** $\log_2 8$.
**3.** $\log_4 16$.
**4.** $\log_3 \frac{1}{3}$.
**5.** $\log_2 \frac{1}{4}$.
**6.** $\log_4 \frac{1}{64}$.
**7.** $\log_5 .2$
**8.** $\log_8 32$.

***68. Logarithm of a Product.** We can now show that Rule V, § 62, holds true *whatever the base*. That is, if $M$ and $N$ are any two numbers, and $a$ the base, then

$$\log_a MN = \log_a M + \log_a N.$$

Proof. Let $x = \log_a M$ and $y = \log_a N$. Then $a^x = M$ and $a^y = N$ (§ 67). Hence $a^x \cdot a^y = MN$, or $a^{x+y} = MN$. But the last equality means that

$$\log_a MN = x + y = \log_a M + \log_a N. \qquad (§ 67)$$

***69. Logarithm of a Quotient.** Rule VII, § 63, holds true *whatever the base*. That is, if $M$ and $N$ are any two numbers, then

$$\log_a (M \div N) = \log_a M - \log_a N.$$

Proof. Let $x = \log_a M$ and $y = \log_a N$. Then $a^x = M$ and $a^y = N$. (§ 67). Hence, $a^x \div a^y = M \div N$, or $a^{x-y} = M \div N$. But the last equality means that

$$\log_a (M \div N) = x - y = \log_a M - \log_a N.$$

***70. Logarithm of a Power of a Number.** Rule IX, § 64, holds true *whatever the base*. That is, if $M$ is any number and $n$ any (positive integral) power, then

$$\log_a M^n = n \log_a M.$$

Proof. Let $x = \log_a M$. Then $a^x = M$ (§ 67) and hence $a^{nx} = M^n$. But the last equality means that

$$\log_a M^n = nx = n \log_a M.$$

***71. Logarithm of a Root of a Number.** Rule XI, § 65, ho *whatever the base.* That is, if $M$ is any number and $n$ any ( integral) root, then

$$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M.$$

PROOF. Let $x = \log_a M$. Then $a^x = M$ (§ 67) and hence ( $M^{1/n}$, or $a^{x/n} = \sqrt[n]{M}$. But the last equality means that

$$\log_a \sqrt[n]{M} = \frac{x}{n} = \frac{1}{n} \log_a M.$$

***72. Summary.** From the results established in §§ 67–71 it that Rules V–XII, §§ 62–65, are not only true when the base is was there taken) but they are true for *any* base. Tables exist for bases other than 10, but we shall not consider them.

**NOTE.** The reason why 1 cannot be used as a base is that 1 power is equal to 1, that is, we cannot get different numbers by 1 to different powers.

***73. Historical Note.** Logarithms were first introduced a ployed for shortening computation by JOHN NAPIER (1550–1 Scotchman. However, he did not use the base 10, this being fir by the English mathematician BRIGGS (1556–1631), who con the first table of *common* logarithms.

***74. Calculating Machines. The Slide-Rule.** Machines hav invented and are now coming into very general use, especially b neers, by which the processes of multiplication, division, invo and evolution can be immediately performed. The construct

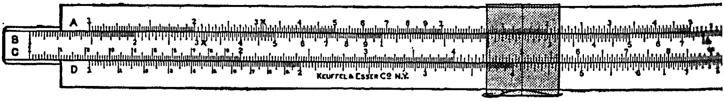

FIG. 38. THE SLIDE RULE

these machines depends upon the principles of logarithms, I describe the machines and their methods of working would t beyond the scope of this text. The simplest machine of this k the *slide rule,* the use of which is easily understood. A simpl rule with directions is inexpensive and may ordinarily be secured booksellers. A full description of the instrument and its use m found in the *Macmillan Tables* (The Macmillan Co., New Yor

# CHAPTER VIII

## COMPOUND INTEREST AND ANNUITIES

**75. Compound Interest.** The interest which $P$ dollars will bring at the end of one year if placed at the rate of interest $i$ is evidently $P \times i$, or $Pi$. If the interest $Pi$ thus received be added to the principal, or $P$, the new principal at the end of the *first* year is $P + Pi$, or

$$P(1+i). \tag{1}$$

If the principal (1) be again allowed to draw interest for one year at the same rate $i$, the interest received will be $P(1+i) \times i$, or $P(1+i)i$, and if this be added (compounded) to the former principal (1), the amount of the principal at the end of the *second* year becomes $P(1+i)+P(1+i)i$, which may be written $P(1+i)(1+i)$, or

$$P(1+i)^2. \tag{2}$$

Similarly, the amount at the end of the *third* year is

$$P(1+i)^3,$$

and, in general, we have the following formula for the amount $A_n$ which will be realized from a principal $P$ by compounding the interest upon it annually for $n$ years at the rate $i$:

$$\boldsymbol{A_n = P(1+i)^n}. \tag{3}$$

EXAMPLE 1. What will be the amount of $225 loaned for 5 years at 8% compound interest?

SOLUTION. Here $P = 225$, $i = .08$ and $n = 5$. Hence, using the formula, we find $A_5 = 225(1+.08)^5 = 225 \times 1.08^5$.

The actual computation of $A$ is now best carried out by logarithms. Thus, taking the logarithm of each member of the last equation, we have, by Rules V and IX, §§ 62, 64,

$$\begin{aligned} \log A_5 &= \log 225 + 5 \log 1.08 = 2.3522 + 5 \times 0.0334 \\ &= 2.3522 + 0.1670 = 2.5192 \end{aligned}$$

Therefore, by § 61, $A_5 = \$330.50$ *Ans.*

EXAMPLE 2. What principal will amount to $1000 in 10 years at 5% compound interest?

SOLUTION. Here $A_{10}=1000$, $P=?$, $i=.05$, $n=10$ so that the formula gives $1000=P(1+.05)^{10}=P(1.05)^{10}$. The problem thus resolves itself into solving this equation for $P$, and this is most readily done by use of logarithms as follows:

$$\log 1000=\log P+10 \log 1.05$$

Hence $\log P=\log 1000-10 \log 1.05=3-0.2120=2.788$ Therefore, by § 61, $P=\$613.70$ *Ans.*

## EXERCISES

**1.** Find the amount of $400 for 10 years at 3% compound interest.

**2.** Find the amount of $100 for 20 years at 6% compound interest.

**3.** What principal loaned at 4% compound interest will amount to $1500 in 10 years?

**4.** What sum of money invested at 4% compound interest from a child's birth until he is 21 years old will yield $1000?

**5.** In what time will $800 amount to $1834.50 if put at compound interest at 5%?

[HINT. Note that the unknown time becomes determined by an exponential equation which can be solved as in § 66.]

**6.** How long will it take a sum of money to double itself at 5% compound interest?

**7.** What is the rate per cent when $300 loaned at compound interest for 6 years will yield $402?

**8.** Solve the formula for $n$ in terms of $A$, $P$, and $i$.

**9.** Construct a graph to show the compound amount of 1 dollar at 6% as the time varies.

**10.** If, instead of the interest being compounded annually as in the formula of § 67, it is compounded $m$ times a year, show that the formula becomes

$$A_{m,n}=P\left(1+\frac{i}{m}\right)^{mn}.$$

**11.** In how many years will $300 amount to $400 at 6% compound interest, the interest being compounded quarterly?

**12.** What sum should be deposited in a bank paying 4% compounded semi-annually in order to discharge a debt of $7430 due ten years later.

**76. Annuities.** An ***annuity*** is a series of equal payments made at equal intervals during a fixed period of time. For convenience, the first payment will here be regarded as made at the *end* of the first *year*, the second payment at the *end* of the second *year*, etc.

Thus, if A has a life insurance policy in the form of an annuity in case of death to B of $1000 a year for 10 years, then at the end of the first year after A's death the company issuing the policy is to pay B $1000, and a like payment is to be made at the end of the second year, third year, etc., up to the end of the tenth year. Evidently, if interest be taken into account, such a policy will be worth more to B than the mere total of $10,000 thus received, since he may during the 10 years be reinvesting the various payments so as to receive additional returns.

The following fundamental general problem thus arises. If we represent the amount of each payment by $a$, the number of yearly payments by $n$ and the interest rate by $i$, what will be the accumulated value $V_n$ of the annuity at the end of the $n$ years? The answer, expressed as a formula for $V_n$ in terms of $a$, $n$ and $i$, is readily obtained as follows.

Using the formula of § 75, we see that the accumulated values of the first, second, $\cdots n$th payments will be:

$$a(1+i)^{n-1},\ a(1+i)^{n-2},\ \cdots,\ a(1+i)^2,\ a(1+i),\ a.$$

The desired value, $V_n$, is therefore the sum of these $n$ expressions. But they are seen to form a geometric progression whose first term is $a$ and whose common ratio is $(1+i)$. The sum is therefore readily expressed by use of the first formula in § 39, which gives

$$V_n = a\frac{(1+i)^n-1}{i}. \tag{4}$$

By the ***present value of an annuity*** of $a$ dollars per annum is meant the amount in cash that one could afford to pay for the privilege of receiving the payments in their regular order. A second fundamental problem thus arises: What is the present value $P$ of an annuity of $a$, payable in $n$ yearly installments when the interest rate is $i$? This again may be answered by simple considerations based on the properties of a geometric progression. Thus, the present value of the *first* payment can be obtained from the formula of § 75 by placing in it $A=a$, $n=1$ and solving for $P$, thus giving $a(1+i)^{-1}$. Similarly, the present value of the second payment is $a(1+i)^{-2}$, that of the $n$th payment being $(1+i)^{-n}$. The desired value of $P$ is therefore the sum of these, or

$$a(1+i)^{-1}+a(1+i)^{-2}+\ \ldots\ +a(1+i)^{-n}.$$

This being a sum of terms forming a geometric progression, its value can be readily expressed as before by the first formula of § 39, which gives as the desired formula

$$P_n=a\frac{1-(1+i)^{-n}}{i}. \tag{5}$$

### EXERCISES

**1.** What will be the accumulated value of an annuity of $100 for 10 years at 6%.

Solution. $V_{10}=\frac{a}{i}[(1+i)^n-1]=\frac{100}{.06}[(1.06)^{10}-1].$

By logarithms, $(1.06)^{10}$ is found to be 1.7904, hence $(1.06)^{10}-1=0.7904$ Therefore

$$\begin{aligned}\log V_{10}&=\log 100+\log 0.7904-\log .06\\&=2+(9.8978-10)-(8.7782-10)=3.1196\end{aligned}$$

Hence $V_{10}=\$1317$, the accumulated value of the annuity. *Ans.*

**2.** What is the present value of an annuity of $300 for 10 years at 6%?

**3.** How much must a man save annually and deposit in a savings and loan company paying 5%, compounded annually, in order to pay off a mortgage of $2000 after 5 years?

**4.** A man buys a house and lot, paying $1500 down and agreeing to pay $1000 annually for the next 4 years. What is the equivalent cash price if money is worth 6% per year?

[Hint. Note that the $1500 payment is not a part of the *annuity*.]

**5.** It is estimated that a certain mine will be exhausted in 10 years. If the mine yields a net annual income of $10,000, what would be a fair purchase price, money being worth 5%?

**6.** Show that if, instead of the installments being made annually, they are made $m$ times a year and the interest compounded at each payment, then the two formulas of § 76, remain the same except that $i/m$ is to be substituted for $i$ and $mn$ for $n$.

**7.** Using the results of Ex. 6, answer the following question: A piano is sold for $100 cash and $50 to be paid semi-annually for 3 years. What is the equivalent cash price, if money is worth 6%, compounded semi-annually?

**8.** A city is to issue 20-year bonds to the amount of $100,000 for the erection of public schools and it is desired to establish a "sinking fund" to provide for the extinction of the debt when due. How much must be deposited in the sinking fund at the end of each year, money being worth 4% and compounded annually?

**9.** A student borrows $500 at the beginning of each of his four college years, agreeing to pay the debt with 6% interest in 4 annual payments, the first being due 5 years after the first $500 was borrowed. What will be the amount of each payment?

**10.** The beneficiary of a $5000 life insurance policy is to receive ten equal annual payments, the first to be made when the policy matures. Find each payment if money is worth 4% compounded annually.

**11.** A debt of $5000 with interest at 6% was contracted on Jan. 1, 1916. On Jan. 1, 1917, $1,000 was paid and on Jan. 1, 1918, $2,000 was paid. It was then arranged to pay the remainder in 3 annual installments, the first of which should be payable Jan. 1, 1919. How much was each installment?

**12.** How much should be invested at the end of each year to discharge a non-interest bearing debt of $10,000 due in 10 years, interest being 4%?

**13.** A bank pays $940 for a non-interest bearing note due in 1 year. If the face of the note is $1000 what rate of interest is involved?

# CHAPTER IX

## MATHEMATICAL INDUCTION—BINOMIAL THEOREM

**77. Mathematical Induction.** The three following purely arithmetic relations are easily seen to be true:

$$1+2=\tfrac{2}{2}(2+1),$$
$$1+2+3=\tfrac{3}{2}(3+1),$$
$$1+2+3+4=\tfrac{4}{2}(4+1).$$

We might at once infer from these that if $n$ be *any* positive integer, there exists the algebraic relation

$$(1)\qquad 1+2+3+4+\cdots+n=\frac{n}{2}(n+1),$$

the dots indicating that the addition of the terms on the left continues up to and including the number $n$.

For example, if $n=8$, this would mean that

$$1+2+3+4+5+6+7+8=\tfrac{8}{2}(8+1).$$

Again, if $n=10$, it would mean that

$$1+2+3+4+5+6+7+8+9+10=\tfrac{10}{2}(10+1).$$

That these are indeed true relations is discovered as soon as we simplify them. Let the pupil convince himself on this point.

It is to be carefully observed, however, that the inference just made, namely that (1) is true for *any* $n$, is not yet justified, for we have only shown that (1) holds good for certain *special* values of $n$, and we could never hope to do more than this however long we continued to try out the formula in this way.

Something more than a knowledge of special cases must always be known before any perfectly certain *general* inference can be made. For example, the fact that Saturday was cloudy for 38 weeks in succession gives no certain information that it will be so on the 39th week.

We shall now show how the general formula (1) may be established free from all objection; that is, in a way that leaves no possible question as to its truth in all cases.

Let $r$ represent any one of the *special* values of $n$ for which we know (1) to be true. Then

$$1+2+3+4+\cdots+r=\frac{r}{2}(r+1). \tag{2}$$

Let us add $(r+1)$ to both sides. The result is

$$1+2+3+4+\cdots+r+(r+1)=\frac{r}{2}(r+1)+(r+1).$$

In the second member of the last equation we may write

$$\frac{r}{2}(r+1)+(r+1)=(r+1)\left(\frac{r}{2}+1\right)=(r+1)\left(\frac{r+2}{2}\right)=\frac{r+1}{2}(r+2).$$

while the first member has the same meaning as

$$1+2+3+\cdots+(r+1).$$

Thus, (2) being given us, it follows that we may write

$$1+2+3+4+\cdots+(r+1)=\frac{r+1}{2}(r+2). \tag{3}$$

But (3) is seen to be precisely the same as (2) except that $r+1$ now replaces $r$ throughout. This means that if (1) is true when $n=r$, as we have supposed, then it holds true *necessarily* for the next greater value of $n$, which is $r+1$.

The original fact which we wished to establish (namely, that (1) is true for *any* $n$) now follows without difficulty. In fact, we know (see beginning of this section) that (1) is true when $n=4$, from which it now follows that it must be true also when $n=5$. Being true when $n=5$, the same reasoning shows that it must be true also when $n=6$. Thus, we may reach any given integer $n$, however large it may be. Hence (1) is true for any such value of $n$.

This method of reasoning illustrates what is termed ***mathematical induction***. Another example of the process will now be given, in a more condensed form.

Example. Prove by mathematical induction that

(1) $1+3+5+7+\cdots+(2n-1)=n^2$. ($n$ = *any positive integer*)

Solution. When $n=1$, the formula gives $1=1^2$; when $n=2$, it gives $1+3=2^2$; when $n=3$, it gives $1+3+5=3^2$, all of which arithmetical relations are seen to be correct.

Let $r$ represent any value of $n$ for which the formula has been proved. Then

(2) $$1+3+5+7+\cdots+(2r-1)=r^2.$$

Adding $(2r+1)$ to each member gives

(3) $$1+3+5+7+\cdots+(2r+1)=r^2+(2r+1)=r^2+2r+1=(r+1)^2.$$

But (3) is the same as (2) except that $r$ has been replaced throughout by $r+1$. Hence, if (1) is true for any value of $n$, such as $r$, it is necessarily true also for that value of $n$ increased by 1.

Now, we know (1) to be true when $n=3$. (See above.) Hence it must be true when $n=4$. Being true when $n=4$, it must be true when $n=5$, etc., and in this way we now know that (1) is true for any value (positive integral) of $n$ whatever.

## EXERCISES

Prove the correctness of each of the following formulas by mathematical induction, $n$ being understood to be any positive integer.

**1.** $2+4+6+8+\cdots+2n=n(n+1)$.

[Hint. First try out for $n=1$, $n=2$, and $n=3$. Let $r$ represent a number for which the formula holds. Add $2(r+1)$ to both members of the resulting equation and compare results.]

**2.** $3+6+9+12+\cdots+3n=\frac{3n}{2}(n+1)$.

**3.** $1^2+2^2+3^2+4^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

**4.** $2^2+4^2+6^2+\cdots+(2n)^2=\frac{2}{3}n(n+1)(2n+1)$.

**5.** $1^3+2^3+3^3+4^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$.

**6.** $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

**7.** $2+2^2+2^3+2^4+\cdots+2^n=2(2^n-1)$.

**8.** Prove that if $n$ is any positive integer, $a^n-b^n$ is divisible by $a-b$.

[Hint. Since $a^{r+1}-b^{r+1}=a(a^r-b^r)+b^r(a-b)$, it follows that $a^{r+1}-b^{r+1}$ will be divisible by $a-b$ whenever $a^r-b^r$ is divisible by $a-b$.]

**9.** Prove that $a^{2n}-b^{2n}$ is divisible by $a+b$.

**78. The Binomial Theorem.** If we raise the binomial $(a+x)$ to the second power, that is, find $(a+x)^2$, the result is $a^2+2ax+x^2$. Similarly, by repeated multiplication of $(a+x)$ into itself, we can find the expanded forms for $(a+x)^3$, $(a+x)^4$, $(a+x)^5$, etc. The results which we find in this way have been placed for reference in a table below:

$$(a+x)^2=a^2+2ax+x^2.$$

$$(a+x)^3=a^3+3a^2x+3ax^2+x^3.$$

$$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$$

$$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

$$(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6, \text{ etc.}$$

Upon comparing these, it appears that the expansion of $(a+x)^n$, *where $n$ is any positive integer,* has the following properties:

1. *The exponent of $a$ in the first term is $n$, and it decreases by 1 in each succeeding term.*

The last term, or $x^n$, may be regarded as $a^0 x^n$ (See § 8).

2. *The first term does not contain $x$. The exponent of $x$ in the second term is 1 and it increases by 1 in each succeeding term until it becomes $n$ in the last term.*

3. *The coefficient of the first term is 1; that of the second term is $n$.*

4. *If the coefficient of any term be multiplied by the exponent of $a$ in that term, and the product be divided by the number of the term, the quotient is the coefficient of the next term.*

For example, the term $6a^2x^2$, which is the *third* term in the expansion of $(a+x)^4$, has a coefficient, namely 6, which may be derived by multiplying the coefficient of the preceding term (which is 4) by the exponent of $a$ in that term (which is 3) and dividing the product thus obtained by the number of that term (which is 2).

5. *The total number of terms in the expansion is $n+1$.*

The results just observed regarding the expansion of $(a+x)^n$, where $n$ is any positive integer, may be summarized and condensed into a single formula as follows:

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+\cdots,$$

the dots indicating that the terms are to be supplied in the manner indicated up to the last one, or $(n+1)$st.

This formula is called the ***binomial theorem.*** By means of it, one may write down at once the expansion of any binomial raised to any positive integral power.

That the formula is true in *all* cases, when $n$ is a positive integer, will be proved in detail in § 80. We assume its truth here for those small values of $n$ for which its correctness is easily tested.

EXAMPLE 1. Expand $(a+x)^6$.

SOLUTION. Here $n=6$, so the formula gives

$$(a+x)^6=a^6+6a^5x+\frac{6\cdot 5}{1\cdot 2}a^4x^2+\frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}a^3x^3+\frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}a^2x^4$$
$$+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}ax^5+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6.$$

Simplifying the various coefficients by performing the possible cancelations in each, we obtain

$$(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6.\quad \textit{Ans.}$$

NOTE. It may be observed that the coefficients of the first and last terms turn out to be the same; likewise the coefficients of the second and next to the last terms are the same, and so on symmetrically as we read the expansion from its two ends.

EXAMPLE 2. Expand $(2-m)^5$.

SOLUTION. Here $a=2$, $x=-m$, and $n=5$. The formula thus gives

$$(2-m)^5=2^5+5\cdot 2^4(-m)+\frac{5\cdot 4}{1\cdot 2}\cdot 2^3(-m)^2+\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\cdot 2^2(-m)^3$$
$$+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}\cdot 2(-m)^4+\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}(-m)^5.$$

Simplifying the coefficients (as in Example 1), this becomes

$$(2-m)^5=2^5+5\cdot 2^4(-m)+10\cdot 2^3(-m)^2+10\cdot 2^2(-m)^3 +5\cdot 2(-m)^4+(-m)^5.$$

Making further simplifications, we obtain

$$(2-m)^5=32-80m+80m^2-40m^3+10m^4-m^5. \quad \textit{Ans.}$$

**NOTE.** The result for $(2-x)^5$ is the same as that for $(2+x)^5$ except that the signs of the terms are alternately positive and negative instead of all positive. A similar remark applies to the expansion of every binomial of the form $(a-x)^n$ as compared to that of $(a+x)^n$.

### EXERCISES

Expand each of the following powers.

**1.** $(x+y)^3$.
**2.** $(a+b)^4$.
**3.** $(x-y)^3$.
**4.** $(a-b)^4$.
**5.** $(2+r)^5$.
**6.** $(a+x)^7$.
**7.** $(y-3)^5$.
**8.** $(a^2+x)^5$.
**9.** $(a^2-x^2)^4$.
**10.** $(2a+1)^4$.
**11.** $(x-3y)^5$.
**12.** $(1+x^2)^6$.
**13.** $(1-x)^8$.
**14.** $(x-\frac{1}{2})^5$.
**15.** $(3a^2-1)^4$.
**16.** $(a+x)^{10}$.
**17.** $\left(\frac{1}{x}+\frac{1}{y}\right)^7$.
**18.** $\left(\frac{a}{x}-\frac{x}{a}\right)^5$.
**19.** $(\sqrt[3]{a^2}+\sqrt[4]{b^3})^3$.
**20.** $\left(\sqrt{2}+\frac{1}{x^2}\right)^3$.

**79. The General Term of $(a+x)^n$.** The third term in the expansion of $(a+x)^n$, as given by the formula in § 78, is

$$\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2. \quad \textit{(third term)}.$$

Observe that the exponent of $x$ is 1 less than the number of the term; the exponent of $a$ is $n$ minus the exponent of $x$; the last factor of the denominator equals the exponent of $x$; in the numerator there are as many factors as in the denominator.

Precisely the same statements can be made as regards the fourth term, or

$$\frac{n(n-1)(n-2)}{1\ \ 2\cdot 3}a^{n-3}x^3. \quad \textit{(fourth term)}.$$

In the same way, it appears that the above statements can be made of *any* term, such as the $r$th, so that the formula for the $r$th term is

$$rth\ term = \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}\cdot a^{n-r+1}x^{r-1}.$$

Example. Find the 7th term of $(2b-c)^{10}$.

Solution. Here

$$a=2b,\quad x=(-c),\quad n=10,\quad \text{and}\quad r=7.$$

Therefore (using the formula), the desired 7th term is

$$\text{Seventh term} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot(2b)^4(-c)^6$$

$$=210(2b)^4(-c)^6=3360b^4c^6.\quad \textit{Ans.}$$

### EXERCISES

Find each of the following indicated terms.

1. 5th term of $(a+x)^8$.
2. 6th term of $(x-y)^8$.
3. 7th term of $(2+x)^9$.
4. 10th term of $(m-n)^{14}$.
5. 6th term of $(a^2-b^2)^{10}$.
6. 20th term of $(1+x)^{24}$.
7. 6th term of $\left(x+\frac{1}{x}\right)^{11}$.
8. 9th term of $\left(\frac{a}{b}-b\right)^{16}$.
9. 5th term of $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^{12}$.
10. 4th term of $(2\sqrt{2}-\sqrt[3]{3})^6$.

**80. Proof of the Binomial Theorem.** The way in which the binomial formula was established in § 78 is, strictly speaking, open to objection because we there made sure of its correctness only for certain special values of $n$, such as $n=2$, $n=3$, $n=4$, and $n=5$. Though the formula holds true, as we saw, in these cases, it does not follow necessarily that it is true in *every* case, that is, for every positive integral value of $n$. We can now establish this fact, however, by the process of mathematical induction, when $n$ is a positive integer.

Let $m$ represent any special value of $n$ for which the formula has been established (as, for example, 2, 3, 4, or 5). Then we have

$$(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{1\cdot 2}a^{m-2}x^2+\cdots$$

$$(1)\qquad +\frac{m(m-1)\cdots(m-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{m-r+1}x^{r-1}+\cdots+x^m.$$

Let us now multiply both members of this equation by $a+x$. On the left we obtain $(a+x)^{m+1}$. On the right we shall have the sum of the two results obtained by multiplying the right side of (1) first by $a$ and then by $x$, that is we shall have the sum of the two following expressions:

$$a^{m+1}+ma^mx+\frac{m(m-1)}{1\cdot 2}a^{m-1}x^2+\cdots$$
$$+\frac{m(m-1)\cdots(m-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{m-r+2}x^{r-1}+\cdots+ax^m,$$

and

$$a^mx+ma^{m-1}x^2+\cdots+\frac{m(m-1)\cdots(m-r+3)}{1\cdot 2\cdot 3\cdots(r-2)}a^{m-r+2}x^{r-1}$$
$$+\cdots+max^m+x^{m+1}.$$

Adding these, and making the natural simplifications in the resulting coefficients of $a^mx$, $a^{m-1}x^2$, etc., and equating the final result to its equal on the left (namely $(a+x)^{m+1}$, as noted above) gives

$$(a+x)^{m+1}=a^{m+1}+(m+1)a^mx+\frac{(m+1)m}{1\cdot 2}a^{m-1}x^2+\cdots$$

$$(2)\qquad +\frac{(m+1)m\cdots(m-r+3)}{1\cdot 2\cdot 3\cdots(r-1)}a^{m-r+2}x^{r-1}+\cdots+x^{m+1}.$$

But (2) is precisely (1) except for the substitution of $m+1$ for $m$ throughout. Hence, if the binomial formula holds for any special value of $n$, as $m$, it necessarily holds for the next larger value, namely $m+1$. But we have already observed that it holds when $n=5$. It must, therefore, hold when

$n=5+1$, or 6. But if it holds when $n=6$, it must likewise hold when $n=6+1$, or 7. Thus we may proceed until we arrive at any chosen value of $n$ whatever. That is, the formula must be true for *any* positive integral value of $n$.

***81. The Binomial Formula for Fractional and Negative Exponents.** In case the exponent $n$ is *not* a positive integer but is fractional or negative, we may still write the expansion of $(a+x)^n$ by the formula of § 78, but it will now contain indefinitely many terms instead of coming to an end at some definite point; that is, we meet with an *infinite series.* (Compare § 41.)

For example,

$$(a+x)^{\frac{1}{2}}=a^{\frac{1}{2}}+\tfrac{1}{2}a^{\frac{1}{2}-1}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}a^{\frac{1}{2}-2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}a^{\frac{1}{2}-3}x^3+\cdots$$

$$=a^{\frac{1}{2}}+\tfrac{1}{2}a^{-\frac{1}{2}}x+\frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}a^{-\frac{3}{2}}x^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}a^{-\frac{5}{2}}x^3+\cdots$$

$$=a^{\frac{1}{2}}+\tfrac{1}{2}a^{-\frac{1}{2}}x-\tfrac{1}{8}a^{-\frac{3}{2}}x^2+\tfrac{1}{16}a^{-\frac{5}{2}}x^3+\cdots$$

Here we have written only the first four terms of the expansion, but we could obtain the 5th term in the same way and as many others in their order as might be desired.

**82. Historical Note.** The binomial formula for cases in which the exponent $n$ is a positive integer was known to the early Greek and Arabic mathematicians, but its significance when $n$ is fractional was first pointed out by *Sir Isaac Newton* (1642–1727).

***83. Application.** If in $(a+x)^n$ the value of $x$ is small in comparison to that of $a$ (more exactly, if the numerical value of $x/a$ is less than 1) then the first few terms of the expansion furnish a close approximation to the value of $(a+x)^n$. This fact is often used to find approximate values for the roots of numbers in the manner illustrated below.

Example. Find the approximate value of $\sqrt{10}$.

Solution. Write $\sqrt{10}=\sqrt{9+1}=\sqrt{(3^2+1)}$ and expand this last form by the binomial formula. Thus (using the final result in the worked example of § 81), we have

$$\sqrt{10}=(3^2+1)^{\frac{1}{2}}=(3^2)^{\frac{1}{2}}+\tfrac{1}{2}(3^2)^{-\frac{1}{2}}\cdot 1-\tfrac{1}{8}(3^2)^{-\frac{3}{2}}\cdot 1^2+\tfrac{1}{16}(3^2)^{-\frac{5}{2}}\cdot 1^3+\cdots$$

$$=3+\frac{1}{2\cdot 3}-\frac{1}{8\cdot 3^3}+\frac{1}{16\cdot 3^5}+\cdots,$$

$$=3+.166666-.004629+.000257=3.162294 \text{ (approximately).}$$

Observe that the value of $\sqrt{10}$ as given in the tables is 3.16228, thus agreeing with that just found so far as the first four places of decimals are concerned.

Whenever extracting roots by this process we use the following general rule.

*Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root, and expand the result by the binomial theorem.*

### *EXERCISES

Write the first four terms in the expansion of each of the following expressions.

**1.** $(a+x)^{\frac{2}{3}}$.

**2.** $(a+x)^{-2}$.

**3.** $(1+x)^{\frac{1}{3}}$.

**4.** $(2-x)^{-\frac{1}{4}}$.

**5.** $(2a+b)^{\frac{3}{4}}$.

**6.** $(a^3-x^2)^{-\frac{3}{4}}$.

**7.** $\sqrt[5]{2+x}$.

**8.** $\sqrt[5]{a+x}$.

**9.** Find by the formula in § 79 the 6th term in the expansion of $(a+x)^{\frac{1}{2}}$.

Find the

**10.** 5th term of $(a+x)^{\frac{1}{2}}$.

**11.** 7th term of $(a+x)^{-\frac{2}{3}}$.

**12.** 8th term of $(1+x)^{\frac{1}{3}}$.

**13.** 9th term of $(a-x)^{-3}$.

**14.** 10th term of $\sqrt{(x+y)^3}$.

**15.** 6th term of $\sqrt[3]{2a+b}$.

Find the approximate values of the following to six decimal places and compare your results for the first three examples with those given in the tables.

**16.** $\sqrt{17}$.

**17.** $\sqrt{27}$.

**18.** $\sqrt[3]{9}$.

**19.** $\sqrt[4]{14}$.

**20.** $\sqrt[5]{35}$.

[Hint. Write $14=16-2=2^4-2$.]

# CHAPTER X

## FUNCTIONS

**84. The Function Idea.** In ordinary speech we make such statements as the following:

1. The area of a circle depends upon its radius.

2. The time it takes to go from one place to another depends upon the distance between them.

3. The power which an engine can exert depends upon the pressure per square inch of the steam in the boiler.

Another way of stating these facts is as follows:

1. The area of a circle is a *function* of its radius.

2. The time it takes to go from one place to another is a *function* of the distance between them.

3. The power which an engine can exert is a *function* of the pressure per square inch of the steam in the boiler.

The idea thus conveyed by the word ***function*** *is that we have one magnitude whose value is determined as soon as we know the value of some other one (or more) magnitudes upon which the first one depends.* This idea is at once seen to be universal in everyday experience and for that reason it becomes of great importance in mathematics.† In the present chapter we shall indicate briefly some of its most essential features, noting especially the significance of the idea when considered graphically.

**85. Types of Algebraic Functions.** An expression of the form

$$(1) \qquad a_0x+a_1,$$

where the coefficients $a_0$ and $a_1$ have any given values (except that $a_0$ must not be 0) is called a ***linear function*** of $x$.

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†The extended formal study of the function idea enters into that branch of mathematics known as the *Calculus*.

Observe that every such expression depends for its value upon the value assigned to $x$, and is determined as soon as $x$ is known. Hence it is a function of $x$ in the sense explained in § 84. It is called a *linear* function since it is of the first degree in $x$. (Compare § 6.)

For example, $2x+3$ is a linear function of $x$. Here we have the form (1) in which $a_0=2$ and $a_1=3$. Similarly, $3x-2$, $x-4$, $-x+\frac{1}{4}$ and $3x$ are linear functions of $x$. (Why?)

Likewise, $3t+2$ is a linear function of $t$, while $-r+5$ is a linear function of $r$, etc.

As an example of a linear function in everyday experience, suppose that in Fig. 39 a person starts from the point $P$ and moves to the right at the rate of 15 miles per hour, and let $Q$ be the point 10 miles to the left of $P$. Then we may say that the distance of the traveler from $Q$ is a linear function of the time he has been traveling, for if $t$ represent the number of hours he has been traveling, his distance from $P$ is $15t$ (see § 7, formula 4) and hence his distance from $Q$ is $15t+10$. This is a linear function of $t$, being of the form (1) in which $a_0=15$ and $a_1=10$.

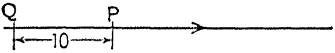

FIG. 39

Likewise, the interest which a given principal, $P$, will yield in one year is a linear function of the rate, for, if $r$ be the rate, the interest in question is given by the formula $P\times r$, or $Pr$, and this is seen to be of the form (1) in which $a_0=P$, and $a_1=0$, $r$ being here the variable.

An expression of the form

$$a_0x^2+a_1x+a_2, \tag{2}$$

where $a_0$, $a_1$, and $a_2$ have any given values (except that $a_0$ must not be 0) is called a ***quadratic function*** of $x$.

For example, $2x^2+3x-1$ is a quadratic function of $x$ because it is of the form (2) in which $a_0=2$, $a_1=3$, $a_2=-1$. Likewise, $x^2+\frac{1}{4}x$; $x^2+\frac{1}{4}$; $-x^2+3x$; $5x^2$; $x^2$ are quadratic functions of $x$.

Again, we may say that the area of a square is a quadratic function of the length of one side, for if $x$ be the length of side, the area is $x^2$ and this is of the form (2) in which $a_0=1$, $a_1=a_2=0$.

Similarly, the area of a circle is a quadratic function of the radius $r$ since it is equal to $\pi r^2$.

An expression of the form

(3) $$a_0x^3+a_1x^2+a_2x+a_3,$$

where $a_0$, $a_1$, $a_2$ and $a_3$ have any given values (except that $a_0$ must not be 0) is called a ***cubic function*** of $x$.

For example, $3x^3-x^2+\frac{1}{2}x-1$; $4x^3-x$; $x^3-2x^2+1$; $5x^3$; $x^3$, etc.

Again, we may say that the volume of a cube is a cubic function of the length of one edge. Also, the volume of a sphere is a cubic function of the radius $r$, since it is equal to $\frac{4}{3}\pi r^3$.

It may now be observed that the expressions (1), (2), and (3) are but special forms of the more general expression

(4) $$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n,$$

where it is understood that $n$ can be any positive integer, while the coefficients $a_0$, $a_1$, $a_2$, $\cdots$ $a_n$ have any given values (except that $a_0$ must not be 0). This is called the ***general integral rational function*** of $x$, or more simply, a ***polynomial*** in $x$. It reduces to the linear function (1) when $n=1$; to the quadratic function (2) when $n=2$; etc.

In addition to these, expressions such as

$$\sqrt{x},\ \sqrt[3]{x},\ \sqrt[4]{x},\ 3\sqrt{x}+\sqrt[5]{x},\ x^2+4x^{-\frac{2}{3}},\quad \frac{5x}{\sqrt[3]{x}-1},$$

and others that involve powers and roots of $x$ may occur in the expression of functions in algebra.

## EXERCISES

**1.** Show that the thickness of a book is a linear function of the number of its pages.

[HINT. Let $x$ be the number of pages, $d$ be the thickness of each page, and $D$ the thickness of each cover. Now build up the formula for the thickness of the book and note which of the functional types in § 85 is present.]

**2.** The supply of gasoline in a tank was very low, its depth being but 1 inch all over the bottom, when it was replenished from a pipe which delivered 3 gallons per minute. Show that the amount in the tank at any moment during the filling was a linear function of the time since the filling began.

**3**. Show that the force which a steam engine has at any moment at its cylinder is a linear function of the area of the piston; also that it is a linear function of the boiler pressure of the steam per square inch.

**4**. A certain room contains a number of 16-candle-power electric lights and a number of Welsbach gas-burners. Show that the amount of illumination at any time is a linear function of the number of electric lights turned on. Is this true regardless of the number of gas-burners already lighted?

**5**. Show that the perimeter of a square is a linear function of the length of one side; also that the circumference of a circle is a linear function of its radius.

**6**. Show that if each side of a square be increased by $x$, the corresponding increase in the area will be a quadratic function of $x$.

[HINT. Let $a$ = the length of one side of the original square. Then the area is $a^2$ and the area of the new square is $(a+x)^2$. Now formulate the expression for the *increase* in area.]

**7**. Show that if the radius of a circle be increased by $x$, the corresponding increase in area will be a quadratic function of $x$.

**8**. Show that if the edge of a cube be increased by $x$ the corresponding increase in volume will be a cubic function of $x$. State and prove the corresponding statement for a sphere.

**9**. Show that if $y$ varies directly as $x$ (see § 48), then $y$ is a linear function of $x$. Is the *converse* of this statement necessarily true; namely, if $y$ is a linear function of $x$, then $y$ varies directly as $x$?

**10**. When $y$ varies as the square of $x$, to which one of the functional types mentioned in § 85 does $y$ belong? Answer the same question when $y$ varies directly as the sum of $x^3$, $5x^2$, and $3x$.

**11**. A certain linear function of $x$ takes the value 5 when $x=1$ and takes the value 8 when $x=2$. Determine the form of the function.

SOLUTION. Since the function is linear, it is of the form $a_0x+a_1$. Since this expression must (by hypothesis) be equal to 5 when $x=1$, we have $a_0\cdot 1+a_1=5$. Likewise, placing $x=2$, gives $a_0\cdot 2+a_1=8$. Solving these two equations for $a_0$ and $a_1$ we obtain $a_0=3$, $a_1=2$. The desired function is therefore $3x+2$. *Ans.*

**12**. A certain linear function of $x$ takes the value 14 when $x=3$, and takes the value $-6$ when $x=-1$. Determine the function.

**13**. A certain quadratic function takes the value 0 when $x=1$, and the value 1 when $x=2$, and the value 4 when $x=3$. Determine completely the form of the function.

**86. Functions Considered Graphically.** By the *graph of a function* is meant the line or curve which results when some letter, as $y$, is placed equal to the function and the graph is drawn of the equation thus obtained. The purpose of the graph is to bring out clearly and quickly to the eye the relation between the given function and the quantity (variable) upon which it depends for its values.

The method of drawing such graphs is precisely the same as that given in § 6 for equations of the first degree, and in § 25, for quadratic equations.

Thus, in order to obtain the graph of the function $x^3$, we place $y=x^3$ and proceed to assign various values to $x$ and compute (from this equation) the corresponding values of $y$, then we plot each point thus obtained and finally draw the smooth curve passing through all such points.

Below is a table of several values of $x$ and $y$ thus computed; and the graph is shown in Fig. 40.

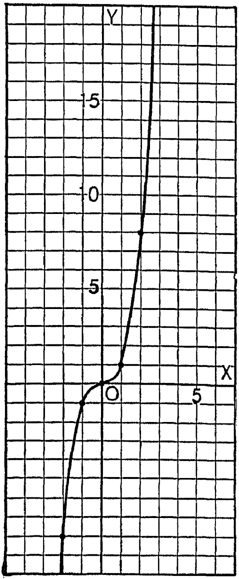

FIG. 40

| When $x=$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| then $y=$ | $-8$ | $-1$ | $0$ | $1$ | $8$ | $27$ | $64$ |

The portion of the curve lying to the right of the $y$-axis extends upward indefinitely, while the portion to the left of the same axis extends downward indefinitely. Note that, from the way this curve has been drawn, it at once brings out to the eye the value of the given function $x^3$ for any value of the letter $x$ upon which this function depends, the function values being the *ordinates* (§ 6) of the points on the curve. For example, at $x=2$ the corresponding ordinate measures 8, which is the function value then present.

This curve may be used as a *graphical table of cubes* of numbers. Thus, if $x = 1.5$, $y = 3.4$, approximately, etc. Likewise, if $y$ is given first, the curve shows the *cube root* of $y$; for example, if $y = 4$, $x$ is about

1.6. The figure may be drawn by the student on a much larger scale; the values of $x$ and $y$ can be read much more accurately from such a figure than from Fig. 40.

Another means of improving the accuracy of the figure is to take a longer distance on the horizontal line to represent one unit than is taken to represent one unit on the vertical scale.

Considering now the various types of functions described in § 85, it is to be noted first that the graph of every *linear* function is a straight line.

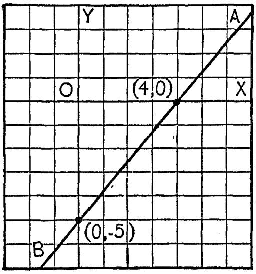

FIG. 41

For example, in considering the graph of the linear function $\frac{5}{4}x-5$, we place $y=\frac{5}{4}x-5$. But this is an equation of the first degree between $x$ and $y$ and hence (§ 6) its graph is a straight line. Fig. 41 shows the result.

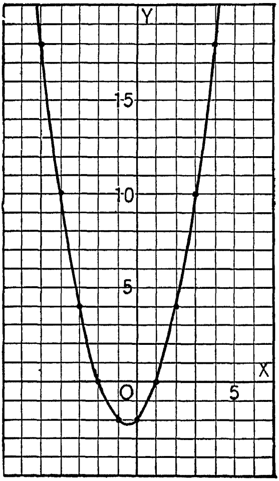

FIG. 42

Note that the graph cuts the $x$-axis in *one* point. The abscissa of this particular point is 4, which indicates that 4 is the root, or solution, of the equation $\frac{5}{4}x-5=0$, for it is this value of $x$ that makes $y=0$.

The graph of every *quadratic* function belongs to the class of curves known as *parabolas*. A parabola resembles in form an oval, open at one end. It never cuts the $x$-axis in more than *two* points. In some cases it lies entirely above or below the $x$-axis, thus not cutting it at all.

Fig. 42 shows the graph of the quadratic function $x^2+x-2$. Note that the curve cuts the $x$-axis at two points whose abscissas are $-2$ and 1, respectively. This indicates that $-2$ and 1 are the roots of the quadratic equation

$$x^2+x-2=0.$$

The general form of the graph of a *cubic* function is that of an indefinitely long smooth curve which cuts the $x$-axis in no more than *three* points.

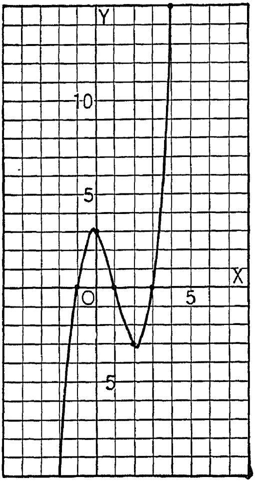

FIG. 43

Fig. 43 shows the graph of the cubic function $x^3-3x^2-x+3$. It cuts the $x$-axis at three points whose abscissas are respectively $-1$, 1, and 3. These values, therefore, are the roots of the cubic equation $x^3-3x^2-x+3=0$.

Similarly, the general form of the graph of the rational integral function of the *fourth* degree is that of an indefinitely long smooth curve which cuts the $x$-axis in no more than *four* points. And it may be said likewise that the graph of the general integral function of degree $n$ (see (4), § 85) is an indefinitely long smooth curve which cuts the $x$-axis in no more than $n$ points.

Fig. 44 shows, for example, the graph of $2x^4-5x^3+5x-2$, this being a function of the fourth degree. The four points where the curve cuts the $x$-axis have abscissas which are equal respectively to $-1$, $\frac{1}{2}$, 1, and 2. These values, therefore, are the roots of the equation $2x^4-5x^3+5x-2=0$.

Fractional expressions give rise to more complex graphs, which may have more than one piece. Fig. 45 shows, for example, the graph of $1/x$. If we let $y=1/x$, *y varies inversely as x* (§ 45). The curve is therefore similar to that drawn in § 50, Fig. 37. The graph consists of two branches and belongs to the class of curves known as *hyperbolas*. These we have already met in § 28.

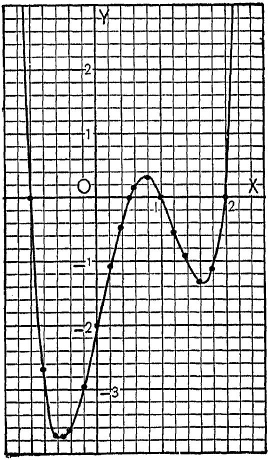

FIG. 44

### EXERCISES

Draw the graphs of the following functions by plotting several points on each and drawing the curve through them. Try to plot enough points so that the form and location of the various waves, or arches, of the curve will be brought out clearly, as in the figures of § 86. Note how many times the curve cuts the $x$-axis and make such inferences as you can regarding the roots of the corresponding equation.

[HINT. When the graph of a quadratic function fails to cut the $x$-axis, this indicates that the roots of the corresponding quadratic equation are imaginary. (See §§ 26–27.) Similarly, when the graph of a cubic function cuts the $x$-axis in but one point, this indicates that there is but one real root to the corresponding equation, the other two roots being imaginary. In general, the number of times the graph cuts the $x$-axis indicates the number of *real* roots of the corresponding equation, the number of imaginary roots being the degree of the equation minus the number of real roots.]

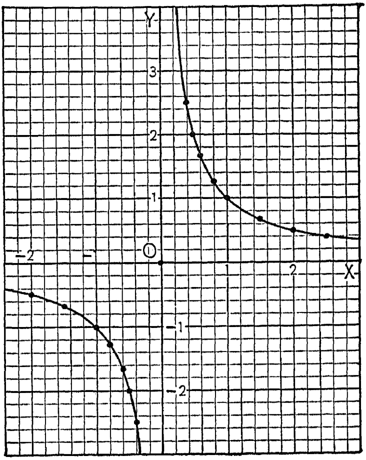

FIG. 45

**1.** $3x+4$. **2.** $x$. **3.** $x^2-x-2$. **4.** $x^2-4$.
**5.** $x^2+1$. **6.** $x^3-3x^2-x+3$. **7.** $x^3+3x^2+2x+6$.
**8.** $x^3-4x$ **9.** $x^4-16$ **10.** $3x^2-x$ **11.** $1/x^2$

**87. The Derivative of a Function.** An examination of the curves shown in Figs. 42–45 shows at once that the *steepness* of any one of them changes from point to point.

For example, in Fig. 42, which is the graph of the function $y=x^2+x-2$, if we select a point on the curve near to its lowest point, the curve is almost horizontal there. At the lowest point itself, where $x=-1/2$, the curve becomes actually horizontal. But if we are at the point whose $x$ is 2 or 3, the steepness is seen to be decidedly greater. In fact, as $x$ increases from the value $x = -1/2$ the steepness also is seen to increase, the curve becoming nearer and nearer vertical. The same is true as $x$ decreases steadily through negative values below $-1/2$.

We shall now show how to obtain an expression that will measure the steepness of a graph at any given point upon it. In Fig. 46, where the curve is the same as in Fig. 42, suppose that $P$ is any given point upon the curve. Draw the short line $PQ$ parallel to the $x$-axis, and at $Q$ erect a perpendicular meeting the curve at $P'$. Then the value of the ratio

$$(1) \qquad \frac{QP'}{PQ}$$

may be taken as a fairly good measure of what we mean by the steepness of the curve at $P$, for it measures fairly well the *rise* $QP'$ in the curve at $P$ *as compared to* the small change $PQ$ in the *horizontal* position of the point.

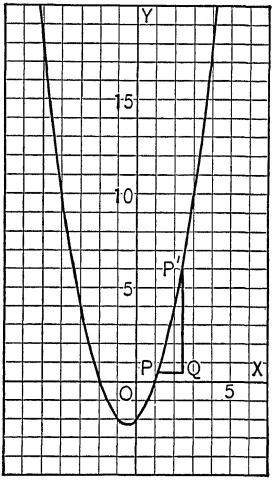

Fig. 46

Thus, the length of $QP'$, as measured on the scale of the drawing, is seen to be about $5\frac{1}{2}$ units, while that of $PQ$ is about $1\frac{1}{2}$ units. The ratio (1) thus becomes $5\frac{1}{2} \div 1\frac{1}{2}$, which reduces to $3\frac{2}{3}$. The steepness at $P$ may therefore be taken as about $3\frac{2}{3}$. If $P$ be selected at a some *higher* elevation on the curve and the corresponding lines $PQ$, $QP'$ be drawn and measured, the ratio (1) will be found to be *greater* than $3\frac{2}{3}$, indicating that the curve is steeper at such a point than at the point $P$ of the drawing.

On the other hand, if $P$ be selected at some elevation *lower* than the one used in the drawing and the same process be carried out, it will turn out that (1) has a value *less* than $3\frac{2}{3}$, indicating less steepness. Evidently, the steepness may be measured, at least roughly, at any point in this manner. It is to be noted, however, that it is essential to the method that $PQ$ be taken small.

Moreover, the smaller $PQ$ be chosen (thus reducing also the length of $QP'$) the closer will the resulting ratio (1) tell the *exact* status of the steepness *at* $P$. Hence, the limit (§ 42) of (1) as $PQ$ is taken closer and closer to zero may be regarded as the exact measure of the steepness at $P$.

Let us now formulate these ideas *algebraically*. Calling $x$ the abscissa of $P$ and letting the small length $PQ$ be represented by $h$, the abscissa of $P'$ will be $x+h$. Since the curve of Fig. 46 is the graph of the equation $y=x^2+x-2$ (see § 86), it follows that the ordinate of $P$ will have the value

$$x^2+x-2 \tag{2}$$

while the ordinate of $P'$ will have the value

$$(x+h)^2+(x+h)-2. \tag{3}$$

Hence, the length of $QP'$, which is the difference of the ordinates of $P'$ and $P$, will be

$$\begin{aligned} QP' &= (x+h)^2+(x+h)-2-(x^2+x-2) \\ &= x^2+2hx+h^2+x+h-2-x^2-x+2 \\ &= 2hx+h^2+h. \end{aligned} \tag{4}$$

Therefore the ratio (1), in the case before us, is given by the formula

$$\frac{QP'}{PQ}=\frac{2hx+h^2+h}{h}, \tag{5}$$

which reduces to

$$\frac{QP'}{PQ}=2x+h+1. \tag{6}$$

The *limit* of this ratio as $PQ$ (or $h$) comes closer and closer to zero is evidently $2x+1$. Hence we arrive at the following conclusion: *If $x$ be the abscissa of a point on the graph of the function* $x^2+x-2$ (Fig. 46), *then the steepness of the curve at that point is equal to* $2x+1$.

Thus, at the point for which $x=1$, the steepness is $2\cdot 1+1$, or 3; at $x=2$, it is $2\cdot 2+1$, or 5; at $x=3$, it is $2\cdot 3+1$, or 7; at $x=0$, it is 1, etc. Note the meaning of these statements in Fig. 46.

It is also to be noted that if $x$ has a value *greater than* $-1/2$ the value of $2x+1$ is *positive*, which indicates that at such a point the curve is *ascending* as $x$ increases. This is illustrated at the point $P$ of Fig. 46. On the other hand, whenever $x$ has a value *less than* $-1/2$, $2x+1$ is *negative*, indicating that at a point corresponding to such a value of $x$ the curve is *descending* as $x$ increases. That this should be so appears directly upon choosing such a point (*i.e.* one for which $x$ is less than $-1/2$) and carrying through the steps of the reasoning on page 149, noting that the expression on the right in (4) will then be necessarily negative, whereas in the case there discussed it was necessarily positive. The reasoning for the new case should be carried through by the student at this point.

Thus, the fact that when $x=-3/2$ we have $2x+1=-2$ indicates that at the point whose $x$ is $-3/2$ the steepness is $-2$ and that the curve (Fig. 46) is *descending* as $x$ increases. Compare with the situation at $x=1/2$.

Similarly, if we start with the function $x^3-3x^2-x+3$ and consider its graph (Fig. 43) we may show by the same process of reasoning that the expression, or formula, determining its steepness from point to point is $3x^2-6x-1$.

In general, the same process enables us to find for any given function a new function which determines for any given $x$ the steepness† of the graph. This new function is called the ***derived function,*** or briefly, the ***derivative*** of the given function.

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†Students familiar with trigonometry will note that what we have defined as the *steepness* of a curve at a point $P$ is equal to the *tangent* of the angle between the tangent line at $P$ and the positive $x$-axis. In fact, the ratio (1) is seen to be equal to the tangent of the angle between $PQ$ and a straight line joining $P$ to $P'$, and as $PQ$ (and hence $QP'$) become smaller, this angle approaches as its limit the angle between the tangent line at $P$ and the positive $x$-axis. In higher mathematics the tangent of this angle is called the *slope* of the tangent line at $P$.

## EXERCISES

**1.** Show (by means of the expression representing the derivative) that the curve in Fig. 46 is twice as steep at the point where $x=5\frac{1}{2}$ as it is at the point where $x=2\frac{1}{2}$.

**2.** Show (using the derivative expression) that the curve in Fig. 46 is three times as steep at the point where $x=-3$ as it is at the point where $x=-1\frac{1}{3}$. Interpret the geometric meaning of the negative signs of the derivative met with in this example.

**3.** Prove the statement (see end of § 87) that the derivative of the function $x^3-3x^2-x+3$ is $3x^2-6x-1$.

[Hint. Take any point $P$ upon the graph shown in Fig. 43 and proceed as in § 87, obtaining an expression analogous to (6) for the ratio (1), and then noting its limit as $h$ approaches zero. It will be necessary first to work out the value for $QP'$ analogous to (4).]

**4.** Using the expression for derivative given in Ex. 3, compare the steepness of the curve in Fig. 43 at the points upon it at which $x=-3$, $-2$, $-1$, 0, 1, 2, 3. Interpret negative results geometrically.

**5.** Prove, following the method of § 87, that the steepness of the graph of the function $\frac{5}{4}x-5$ is everywhere the same, and explain how this result is illustrated in Fig. 41.

**6.** Find (as in § 87) the derivative of the function $2x^4-5x^3+5x-2$. (For the graph, see Fig. 44).

**7.** Find the coordinates of the point upon the graph of

$$y=x^2-4x+1$$

at which the ordinate is *increasing* twice as fast as the abscissa as one passes along the curve from left to right.

**8.** Work Ex. 7 in case the ordinate is to be *decreasing* twice as fast as the abscissa.

**9.** Find the coordinates of the points upon the graph of

$$y=\tfrac{1}{3}x^3+\tfrac{3}{2}x^2+4x$$

at which the steepness is twice as great as at the origin. Draw a figure to illustrate your results.

**10.** Determine the quadratic function of $x$ whose graph passes through the origin and the point (2, 1) and is twice as steep at the latter point as at the former.

**88. Derivative of the General Polynomial.** The derivatives thus far considered have been of certain particular functions forming special cases of the general polynomial mentioned in § 85, that is, of functions of the type form

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n, \tag{1}$$

where $n$ is a positive integer and $a_0$, $a_1$, $a_2$, $\cdots a_n$ are given coefficients. Instead of working out the derivative of each special function as required, it is preferable to work out once for all the expression for the derivative of this *general* function (1). We shall then be able to write down the derivative of any *special* function immediately, saving much labor.

Supposing the graph of (1) to have been drawn, select any point $P$ upon it and let its abscissa be $x$. Then, as in § 87, let $x$ increase by a slight amount, $h$. The ordinate of the first point will have the length (compare (2), § 87)

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n \tag{2}$$

while the ordinate corresponding to the point $x+h$ will have the length (compare (3), § 87)

$$a_0(x+h)^n+a_1(x+h)^{n-1}+a_2(x+h)^{n-2}+\cdots+a_{n-1}(x+h)+a_n. \tag{3}$$

We must now subtract expression (2) from expression (3) (compare (4), § 87). In order to do this, it is desirable first to expand the terms $(x+h)^n$, $(x+h)^{n-1}$, $(x+h)^{n-2}$, etc., by the binomial theorem (§ 78). After we have done so in (3) and have subtracted (2) from the result, all the terms of (2) cancel with like terms in the expanded form of (3), leaving the following expression (compare (4), § 87):

$$\begin{aligned}
&h\left\{na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\ldots+a_{n-1}\right\}\\
&+\frac{h^2}{1\cdot 2}\left\{n(n-1)a_0x^{n-2}+(n-1)(n-2)a_1x^{n-3}+\cdots\right\}\\
&+\ .\ .\ .\\
&+\frac{h^n}{1\cdot 2\cdot 3\cdots n}\left\{n(n-1)(n-2)\cdots 1\right\}a_0.
\end{aligned} \tag{4}$$

It only remains to divide this expression by $h$ and determine the limit approached by the quotient as $h$ approaches zero (compare (5) and (6), § 87). Evidently upon dividing (4) by $h$ we obtain

$$(5)\qquad na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\cdots+a_{n-1}+R,$$

where $R$ contains $h$ as a factor and therefore approaches zero as its limit, so that we reach the following theorem.

THEOREM. *The derivative of the polynomial*

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$$

*is*

$$na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\cdots+2a_{n-2}x+a_{n-1}.$$

An examination of this result shows that the derivative of any polynomial (1) may be immediately written down in accordance with the following rule.

RULE FOR DETERMINING THE DERIVATIVE OF A POLYNOMIAL. *Multiply each term by the exponent of $x$ in that term and diminish the exponent of $x$ by unity.*

Thus the derivative of $2x^3-3x^2+5x-1$ is $2\cdot 3x^2-3\cdot 2x+5$, or $6x^2-6x+5$. Similarly, the derivative of $x^4+3x^3-x^2+2x+3$ is $4x^3+9x^2-2x+2$.

## EXERCISES

Obtain, by use of the Rule in § 88, the derivative of each of the following functions.

**1.** $x^2-3x+2$.

**2.** $5x+1$.

**3.** $x^3+x^2+x+1$.

**4.** $3x^4-4x^3+x$.

**5.** $x^5+3x^4-2x^3+4x^2-x+3$.

**6.** $x^m+x^p$.

**7.** $3x^{2m}+2x^m+1$.

**8.** $x^{p/2}+3x^p+x^{p-1}$.

**9.** Prove that if any polynomial be multiplied by a constant, its derivative will be multiplied by the same constant.

**10.** Prove that the derivative of any constant is equal to zero.

**11.** Show that the graph of $\frac{3}{16}x^4-\frac{7}{12}x^3+\frac{1}{2}x^2+x-1$ is twice as steep when $x=2$ as when $x=1$.

**89. Maxima and Minima Points of the Graph of a Function.** It was shown in § 87 that whenever a value of $x$ renders the derivative *positive*, the graph of the corresponding function, considered at the point having this value of $x$ as its abscissa, will be *ascending* as $x$ increases. Similarly, it was shown that if the derivative has a *negative* value, the graph at the point in question will be *descending* as $x$ increases. It follows that if $x$ be so chosen that the derivative is *equal to zero*, thus being neither positive nor negative, then at the corresponding point on the graph the curve will be neither ascending nor descending; that is, its direction will be horizontal. At such a point (or points) the graph may be either at a highest point or a lowest point of one of its arches, as illustrated at the points $A$, $B$, $C$, $D$, $E$ in Fig. 47. In the former case; that is, at points such as $A$, $C$, $E$, the graph is said to attain a ***maximum,*** while in the latter case, that is, at such points as $B$, $D$ the graph is said to attain a ***minimum.*** Points such as $A$, $C$, $E$ are called *maximum points* of the graph, while points such as $B$, $D$ are its *minimum points.* The points at which the derivative of a function equals zero are called the ***critical points*** of its graph. A quadratic function has *one* critical point, a cubic function has *two* such points, etc. See Figs. 42, 43.

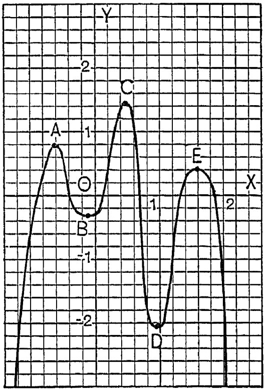

FIG. 47

In summary, then, we have the following result: *The values of $x$ at which the derivative of a function vanishes (equals zero) are the abscissas of the critical points of its graph; the function may be at a maximum or at a minimum at any one of these points.*

The value of this result in the graphical study of functions is illustrated by the following example.

Example. Determine the critical points of the graph of the function

(1) $$y=\tfrac{18}{25}(x^3+x^2-\tfrac{7}{4}x+\tfrac{1}{2}).$$

Solution. The derivative of this function, as immediately written down by the Rule of § 88, is

(2) $$\tfrac{18}{25}(3x^2+2x-\tfrac{7}{4}).$$

The values of $x$ for which this expression vanishes are the roots of the quadratic equation $3x^2+2x-\frac{7}{4}=0$, or, clearing of fractions,

(3) $$12x^2+8x-7=0.$$

Solving the quadratic equation (3) by any one of the usual methods, its roots are found to be $x=\frac{1}{2}$ and $x=-1\frac{1}{6}$.

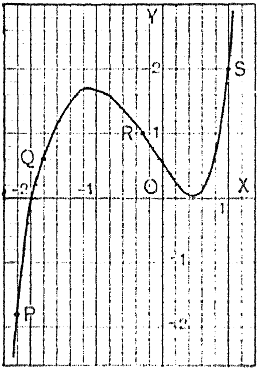

FIG. 48

Therefore, according to the result in § 89, we may say that the *abscissas* of the critical points of the graph of (1) are $x=\frac{1}{2}$ and $x=-1\frac{1}{6}$. To find the *ordinates* of the same points we need only substitute these values of $x$ in (1) to determine the corresponding values of $y$. Thus we find that when $x=\frac{1}{2}$, $y=0$ and when $x=-1\frac{1}{6}$, $y=1\frac{2}{3}$.

The desired critical points of the graph of (1) are therefore the two points whose coordinates are respectively $(\frac{1}{2}, 0)$ and $(-1\frac{1}{6}, 1\frac{2}{3})$. Note how this fact is illustrated in Fig. 48, where the graph of (1) is shown.

The student should observe that as soon as the location of the critical points of a graph are known, the essential character of the graph is determined and the curve can be at once sketched with good approximation, thus avoiding the laborious work of plotting a large number of points.

Thus, in the Example above, when once it was ascertained that the critical points were located at $(\frac{1}{2}, 0)$ and $(-1\frac{1}{6}, 1\frac{2}{3})$, the curve in Fig. 48 could be sketched, at least in its essential form and character. Added accuracy in the drawing could then be obtained by plotting (as in § 25) a few individual points, such as $P$, $Q$, $R$, $S$, and shaping the curve so as to pass through them also.

In Fig. 48 the $x$-axis is a tangent line to the curve.

### EXERCISES

**1.** Prove (by the result in § 89) that the lowest point of the curve in Fig. 42 has its abscissa equal to $-1/2$. What, therefore, is its ordinate?

**2.** Prove that the two critical points of the curve in Fig. 43 have as their abscissas $x=1\pm\frac{2}{3}\sqrt{3}$, and find these values approximately by use of the tables.

**3.** Sketch the graphs of each of the following functions by first locating the critical points of each. (See the Example worked in § 89.)

(*a*) $x^2-x+1$.
(*b*) $\frac{1}{2}x^2-3x$.
(*c*) $x^2-8x+20$.
(*d*) $x^2-8x+16$.
(*e*) $3-2x-x^2$.
(*f*) $\frac{1}{3}x^3+3x^2+8x+1$.
(*g*) $x^3-7x+6$.
(*h*) $x^3-6x^2+11x-6$.

**90. Further Applications of the Derivative.** Aside from the applications which may be made of the derivative of a function in drawing its graph, as described in § 89, there are many other applications related at once to geometry, physics, engineering, etc. This will be best understood from an example.

EXAMPLE. Of all rectangles having a perimeter of 10 inches, which one has the greatest area?

SOLUTION. Let $x$ represent the *length* of any rectangle having a perimeter of 10 inches. Then the *breadth* will evidently be $\frac{1}{2}(10-2x)$, or $5-x$, and hence the *area* will be the product

$$(1) \qquad x(5-x), \text{ or } 5x-x^2.$$

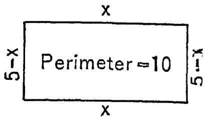

FIG. 49

As thus formulated, the area is clearly a function of $x$, and the problem becomes that of determining the special value of $x$ that will give this function its greatest, or maximum, value. To determine this value of $x$ we now proceed as in § 89.

Finding (by the result in § 88) the derivative of (1) and placing it equal to zero, we have the equation $5-2x=0$, the solution of which is $x=2\frac{1}{2}$.

Therefore, by § 89, the area (1) will be a maximum when $x=2\frac{1}{2}$ inches, which means (see Fig. 49) that the rectangle must be a *square. Ans.*

NOTE. That $x=2\frac{1}{2}$ gives a maximum rather than a minimum appears directly upon drawing the graph of (1).

## APPLIED PROBLEMS

In each of these exercises first formulate, as a function of some suitable variable $x$, an expression for that which is to be made a maximum or a minimum. Proceed as in the solution of the Example in § 90.

**1**. Divide 15 into two parts such that their product is a maximum.

**2**. Divide $h$ into two parts such that the sum of their squares is a minimum.

**3**. Find the number that exceeds its square by the greatest possible amount.

**4**. A garden plot is to be fenced off alongside of a house, using 32 feet of wire fence. What should be the dimensions used in order that the enclosed area shall be the greatest possible.

**5**. It is desired to make an open-top box of greatest possible volume from a square piece of tin whose side is $a$ by cutting equal small squares out of each corner and then folding up the tin to form the sides. What should be the length of a side of the squares cut out?

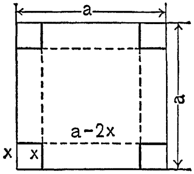

FIG. 50

**6**. A rectangular piece of ground is to be fenced off and divided into three equal parts by fences parallel to one of the sides. What should the dimensions be in order that as much ground as possible may be enclosed with 160 rods of fence?

**7**. The strength of a beam having a rectangular cross section varies jointly as its breadth and the square of its depth. What are the dimensions of the strongest beam that can be sawed out of a round log whose diameter is 14 inches?

**8**. Show that the altitude of the cone of maximum volume that can be inscribed in a sphere of radius $r$ is $\frac{4}{3}r$.

[HINT. Volume of cone $=\frac{1}{3}\times$ *area of base* $\times$ *altitude* $=\frac{1}{3}\pi\overline{AC}^2x$. But, $DAB$ being a right angled triangle, we have

$$\overline{AC}^2=BC\times CD=x(2r-x).$$

Therefore, the volume of the inscribed cone, expressed as a function of its altitude $x$, is

$$\frac{\pi}{3}x^2(2r-x).]$$

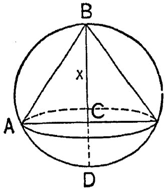

FIG. 51

**9.** Prove that a window of the shape here shown (Norman window) and having a given perimeter, $p$, will admit the most light when the height of its rectangular base equals the radius of its semicircular top.

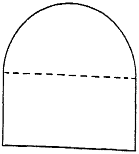

FIG. 52

**10.** Prove that the altitude of the cylinder of maximum volume that can be inscribed in a given right cone is equal to one-third the altitude of the cone.

[HINT. Determine $DG$ in terms of $x$, $h$ and $r$ by making use of the fact that the triangles $BGD$ and $BCA$ are similar. Then express the volume of the cylinder by formula 9, § 7, and find the value of $x$ for which it is a maximum.]

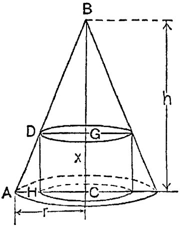

FIG. 53

**11.** A length of wire is cut into two portions which are bent into the shapes of a circle and a square respectively. Show that if the sum of the areas obtained in the two figures is the least possible, the side of the square will be equal to the diameter of the circle.

**91. The Further Study of Functions.** The studies of the present chapter have been confined for the most part to functions of the simplest type, namely, the type of the general rational integral function (4) of § 85. It should be understood, however, that the method explained in § 87 for finding the derivative may be applied to other extended classes of functions also, leading to results which are interesting graphically and of great importance in their applications. For example, one may consider in this way such functions as the following: $1/(1-x)$, $\sqrt{x}$, $10^x$, $\log x$, or in fact, any expression containing the variable $x$. The extended study of this subject belongs to the branch of mathematics known as the *Calculus*.

# CHAPTER XI

## THE THEORY OF EQUATIONS

**92. Introduction.** In Chapter IX it was pointed out that if one draws the graph of any polynomial of $x$, that is, of any function of the type form

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n, \tag{1}$$

where $n$ is a positive integer, the abscissas of the points where the graph cuts the $x$-axis will be the roots (or solutions) of the corresponding equation, namely of the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0. \tag{2}$$

For example, Fig. 43 (page 146) which is the graph of the function $x^3-3x^2-x+3$, brings out the fact that the roots of the equation $x^3-3x^2-x+3=0$ are $-1$, 1 and 3.

This graphical method of determining the roots of an equation cannot ordinarily be relied upon, however, when it is desired to determine the roots *accurately*, since measurements on any drawing, however perfect, are subject to certain inaccuracies of instruments and of eyesight. If the roots are to be determined *exactly*, or at least to any desired degree of accuracy, it is necessary to employ certain special theorems and processes of algebra. These will be considered in the present chapter, together with certain other facts of general interest regarding equations of higher degree than the second.

We shall assume throughout that every equation (2) of the $n$th degree has $n$ roots, and no more, as was indicated in § 86†. In saying this, it is to be understood that both real and imaginary roots are being counted; also that double roots, though equal, are counted as *two*, triple roots as *three*, etc. Compare § 22.

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†This fact may be actually proved, but the proof lies beyond the scope of the present book.

## I. PRELIMINARY THEOREMS

**93. The Remainder Theorem.** For convenience, let us represent the general polynomial with which we are to deal by the symbol $f(x)$, called "function of $x$" or more briefly "$f$ of $x$." That is, let us place

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n.$$

We may then state the following theorem regarding $f(x)$, it being understood that the letter $r$ used below represents any given number.

REMAINDER THEOREM. *If $f(x)$ is divided by $x-r$, the remainder is $f(r)$, where $f(r)$ indicates the value of $f(x)$ when $r$ is substituted for $x$.*

For example, if $2x^3-x^2+2x-1$ (which is a special $f(x)$) be divided in the usual manner by $x-1$, the quotient will be found to be $2x^2+x+3$ with a remainder of 2, that is, we have

$$\frac{2x^3-x^2+2x-1}{x-1}=(2x^2+x+3)+\frac{2}{x-1}.$$

The above theorem says that this remainder, 2, is the same as the result obtained by placing $x=1$ in $2x^3-x^2+2x-1$, that is, the same as $2\cdot1^3-1^2+2\cdot1-1$. The correctness of the statement may be verified immediately.

The student is advised to check the theorem at once in several other similar instances, such as in dividing $3x^3-2x^2+x+1$ by $x-2$, or $x^4+3x^3-2x^2+x-1$ by $x+2$. In the latter case, $r=-2$.

PROOF OF THE REMAINDER THEOREM. We have

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$$

and

$$f(r)=a_0r^n+a_1r^{n-1}+\cdots+a_{n-1}r+a_n.$$

Hence

$$(1)\qquad f(x)-f(r)=a_0(x^n-r^n)+a_1(x^{n-1}-r^{n-1})+\cdots+a_{n-1}(x-r).$$

Since each of the expressions $(x^n-r^n)$, $(x^{n-1}-r^{n-1})$, ... $(x-r)$ is exactly divisible by $(x-r)$ (see Ex. 8, page 132), it follows that the entire rignt hand side of (1) is exactly divisible by $(x-r)$. For brevity, let us indicate the quotient thus obtained by $Q(x)$. We then have

$$\frac{f(x)-f(r)}{x-r}=Q(x), \tag{2}$$

where (since the division is exact) $Q(x)$ is itself a polynomial, but of degree one less than that of $f(x)$, that is, $n-1$.

But the relation (2) may be written in the form

$$\frac{f(x)}{x-r}=Q(x)+\frac{f(r)}{x-r}$$

which states, as desired, that $f(r)$ is the remainder obtained when $f(x)$ is divided by $x-r$.

### EXERCISES

In the following exercises, obtain the answer by means of the remainder theorem, checking its correctness in the first three exercises by long division as in elementary algebra.

Find the remainder when

**1.** $3x^3+x^2-4x+1$ is divided by $x-2$.

**2.** $3x^3+x^2-4x+1$ is divided by $x+2$.

**3.** $x^4+x^3-2x^2+3$ is divided by $x+1$.

**4.** $x^4-3x^2+2$ is divided by $x-3$.

**5.** $ax^2+bx+c$ is divided by $x-h$.

**6.** Prove, by the remainder theorem, that when $2x^4-11x^3+13x^2-3x-4$ is divided by $x-4$ the division is *exact*; that is, the remainder is zero.

**7.** Prove, by the remainder theorem, that

(*a*) $x^n-a^n$ is exactly divisible by $x-a$ for any positive, integral value of $n$.

(*b*) $x^n+a^n$ is exactly divisible by $x+a$ in case $n$ is any *odd* integer. Test also the truth of the statement in case $n$ is any *even* integer.

**94. Synthetic Division.** If it is desired in one of the cases of division considered in § 93 to find not merely the value of the remainder, but also the form of the quotient, the labor of doing so may be very much simplified by following an abridged method known as ***synthetic division.***

Suppose, for example, that it is desired to divide the expression $2x^3-x^2+2x+1$ by $x-1$. By the ordinary long division method, the process would be as follows:

$$
\begin{array}{rrrrl}
2x^3 & -\;x^2 & +2x & +1 & \big|\, \underline{x-1} \\
\underline{2x^3} & \underline{-2x^2} & & & \;2x^2+x+3=\textit{Quotient} \\
& x^2 & +2x & & \\
& \underline{x^2} & \underline{-\;x} & & \\
& & 3x & +1 & \\
& & \underline{3x} & \underline{-3} & \\
& & & +4 & =\textit{Remainder.}
\end{array}
$$

As a first step at simplification, we may evidently concern ourselves only with the coefficients, since, if we knew the coefficients of the quotient to be 2, +1, 3 we could at once supply the needed powers of $x$, obtaining $2x^2+x+3$. This reduces the process to the following form:

$$
\begin{array}{rrrrl}
\mathbf{2} & -1 & +2 & +1 & \big|\, \underline{1-1\qquad} \\
\underline{2} & \underline{-2} & & & \;2+1+3 \\
& \mathbf{+1} & +2 & & \\
& \underline{1} & \underline{-1} & & \\
& & \mathbf{+3} & +1 & \\
& & \underline{3} & \underline{-3} & \\
& & & +4 & =\textit{Remainder}
\end{array}
$$

The numbers in bold type are the same as the coefficients of the quotient, hence the latter may be dispensed with. Moreover, the +2 in the third line of the process and the +1 in the fifth line are mere repetitions of the numbers directly above them in the dividend, hence they may likewise be dispensed with, as also the 2, 1, 3 which appear directly beneath the bold-faced numbers, being mere repetitions of the latter. Thus the process *in its essentials* is as shown below.

$$
\begin{array}{rrrrl}
\mathbf{2} & -1 & +2 & +1 & \big|\, \underline{1-1} \\
& \underline{-2} & \underline{-1} & \underline{-3} & \\
& \mathbf{+1} & \mathbf{+3} & +4 &
\end{array}
$$

But, inasmuch as the divisors which we are considering (see § 93) are always of the simple form $x-r$, the coefficient of $x$ in the divisor is always 1. Hence, in the above process, this 1 may be suppressed, thus replacing $\underline{|\,1-1}$ by $\underline{|\,-1}$; and the work may be written as follows.

$$\begin{array}{r|l} 2-1+2+1 & \underline{-1} \\ -2-1-3 & \\ \hline \mathbf{+1+3+4} & \end{array}$$

Finally, in order to reduce the process to the easiest form for work, we may replace the $\underline{|\,-1}$ by $\underline{|\,+1}$ and *add* throughout the resulting process instead of subtracting, as follows.

$$\begin{array}{r|l} 2-1+2+1 & \underline{+1} \\ +2+1+3 & \\ \hline \mathbf{2+1+3+4} & \end{array}$$

Thus, the quotient is read off as $2x^2+x+3$ and the remainder as 4. Similarly we have the following rule.

RULE FOR SYNTHETIC DIVISION. *To divide $f(x)$ by $x-r$ arrange $f(x)$ in descending powers of $x$, supplying all missing powers by using zeros as their coefficients.*

*Detach the coefficients, writing them horizontally in the order* $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$.

*Bring down the first coefficient $a_0$, multiply it by $r$ and add the result to $a_1$; multiply this sum by $r$ and add the result to $a_2$. Continue this process. The last sum will be the remainder and the preceding sums in their order from left to right will be the coefficients of the various powers of $x$, arranged in descending order, of the quotient.*

Thus, in dividing $x^4-7x^2-5$ by $x-3$, we first write $x^4-7x^2-5$ in the form $x^4+0\cdot x^3-7x^2+0\cdot x-5$. The work of division is then as follows.

$$\begin{array}{r|l} 1+0-7+0-\ 5 & \underline{3} \\ +3+9+6+18 & \\ \hline \mathbf{1+3+2+6+13} & \end{array}$$

Hence, the quotient is $x^3+3x^2+2x+6$, and the remainder is 13.

### EXERCISES

In each of the following exercises, find the value of the quotient and remainder by synthetic division.

**1.** $x^3-4x^2+3x-1$ divided by $x-2$.

**2.** $x^3-4x^2+3x-1$ divided by $x+2$.

**3.** $3x^4+x+1$ divided by $x+1$.

**4.** $x^4+x^3-3x^2-17x-30$ divided by $x+2$.

**5.** $ax^2+bx+c$ divided by $x-h$.

**95. Solutions by Trial, Depressed Equations.** The results indicated in §§ 93, 94 afford a rapid way of determining whether a given number is a root of any given equation $f(x)=0$.

Example 1. Determine whether 6 is a root of the equation

$$2x^4-3x^3-50x^2-27x+10=0.$$

Solution. The result of placing $x=6$ in the first member is (by § 93) equal to the remainder obtained by dividing it by $x-6$, and this remainder, as indicated by the work below, turns out to be $-8$.

$$\begin{array}{rrrrr|l} 2 & -\ 3 & -50 & -27 & +10 & \underline{6} \\ & +12 & +54 & +24 & -18 & \\ \hline 2 & +\ 9 & +\ 4 & -\ 3 & -\ \mathbf{8} & \end{array}$$

Thus, when $x=6$ the first member of the given equation is *not* zero (as the equation requires), but $-8$. We therefore conclude that 6 is *not* a root.

Example 2. Determine whether 4 is a root of the equation

$$x^3-x^2-11x-4=0.$$

Solution. The work in brief is as follows:

$$\begin{array}{rrrr|l} 1 & -1 & -11 & -4 & \underline{4} \\ & +4 & +12 & +4 & \\ \hline 1 & +3 & +\ 1 & \mathbf{+0} & \end{array}$$

The remainder being zero, it follows that 4 *is* a root.

The solution of Example 2 indicates not only that 4 is a root of the given equation

$$(1) \qquad x^3-x^2-11x-4=0,$$

but also that the *quotient* obtained by dividing the first member by $x-4$ is $x^2+3x+1$. Hence, (1) may be written in the form

$$(x-4)(x^2+3x+1)=0,$$

from which it follows (§ 16) that, aside from the root 4 already obtained, the remaining roots of (1) are those of the simpler equation

$$x^2+3x+1=0.$$

Whenever a new equation is thus obtained from an original one through a knowledge of one of its roots, the new equation (whose degree is one lower than the original) is known as the *depressed equation* corresponding to that root. Evidently, whenever a depressed equation can be substituted in this way for an original, the process of determining solutions by trial becomes simplified, and in some cases it leads directly to a determination of *all* the roots of the original equation, as illustrated in the following example.

EXAMPLE 3. Obtain, by the method of trial and the use of depressed equations, such information as is available concerning the integral roots of the equation

$$x^4-2x^3-20x^2-21x-18=0. \tag{2}$$

SOLUTION. Upon performing the tests such as indicated in Examples 1 and 2, with $x=1, 2, 3, 4, 5, 6$, we find that the remainder in each case is *not* zero, except for $x=6$, the work for this case appearing below.

$$\begin{array}{r|l} 1-2-20-21-18 & 6 \\ \hline +6+24+24+18 & \\ \hline 1+4+\ \ 4+\ \ 3+\ \mathbf{0} & \end{array}$$

The depressed equation corresponding to the root 6 is therefore

$$x^3+4x^2+4x+3=0.$$

Testing this equation for $x=1, 2, 3$, etc., we find that the remainders steadily increase. This indicates that the equation has no positive integral root. We proceed, therefore, to test for the *negative* integers $-1$, $-2$, $-3$, etc. It thus appears that $-3$ gives a zero remainder, as shown below.

$$\begin{array}{r|l} 1+4+4+3 & -3 \\ \hline -3-3-3 & \\ \hline 1+1+1+\mathbf{0} & \end{array}$$

The corresponding depressed equation is $x^2+x+1=0$, and this, being a quadratic equation, may be solved by formula (§ 21). Its roots are thus found to be $\frac{1}{2}(-1\pm\sqrt{-3})$. They are therefore imaginary (§ 10). In summary, therefore, equation (2) has the two *real* roots 6, $-3$ and the two *imaginary* roots $\frac{1}{2}(-1\pm\sqrt{-3})$.

### EXERCISES

Obtain, by the methods of § 95, such information as is available regarding the *integral* roots of each of the following equations. If a depressed equation of the *second* degree is finally obtained, solve it, as in Example 3, § 95, thus obtaining *all* the roots of the given equation.

**1.** $2x^3+3x^2-11x-6=0$.

**2.** $2x^3-5x^2-11x-4=0$.

**3.** $x^3-x^2-19x-5=0$.

**4.** $x^4+2x^3-3x^2-8x-4=0$.

**5.** $3x^4-21x^3+22x^2+37x+15=0$.

**6.** $x^4-4x^3+11x-6=0$.

**7.** If $r$ is a root of the cubic equation $ax^3+bx^2+cx+d=0$, determine the corresponding depressed equation.

**96. Transformations of Equations.** The determination of the roots of a given equation is frequently facilitated by transferring its study to that of a related, or transformed equation. In this connection, the theorems stated below are especially important, as will be seen in §§ 98, 99.

THEOREM 1. *Having given an equation of the form*

$$(1)\qquad a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0,$$

*one may obtain an equation each of whose roots is m times the corresponding root of* (1) *as follows. Multiply the successive coefficients of* (1), *beginning with that of* $x^{n-1}$, *by* $m, m^2, m^3, \cdots$ *respectively; in other words, build up the following new (transformed) equation:*

$$(2)\qquad a_0x^n+ma_1x^{n-1}+m^2a_2x^{n-2}+\cdots+m^{n-1}a_{n-1}x+m^na_n=0.$$

Thus, whatever the roots of the equation $3x^3-2x^2+x-4=0$ may be, the roots of the equation $3x^3-2\cdot 2x^2+2^2x-2^3\cdot 4=0$, or $3x^3-4x^2+4x-32=0$, are *twice* as great.

The transformed equation (2) may be obtained at once from (1) by multiplying the respective terms of (1), beginning with the term $a_1x^{n-1}$, by $m, m^2, m^3, \cdots m^n$. It should be noted, however, that in applying this process to a given equation (1), all missing terms are to be supplied with *zero* coefficients.

Thus, in order to obtain the equation whose roots are *three* times the roots of the equation $x^4-2x^2+x-1=0$, one proceeds as follows. Supplying the missing coefficient, we may write the given equation in the form $x^4+0\cdot x^3-2x^2+x-1=0$. Hence, by Theorem 1, the desired equation is $x^4+3\cdot 0\cdot x^3-2\cdot 3^2x^2+3^3\cdot x-3^4=0$, which reduces to $x^4-18x^2+27x-81=0$. *Ans.*

PROOF OF THEOREM 1. What we are to prove may be stated as follows. If $r$ be any root of (2), then the quantity $s=r/m$ will be a root of (1). This, in fact, means that any root of (2) is $m$ times a corresponding root of (1).

Since $r$ is a root of (2) we have

$$(3) \quad a_0r^n+ma_1r^{n-1}+m^2a_2r^{n-2}+\cdots+m^{n-1}a_{n-1}r+m^na_n=0.$$

Substituting for $r$ its value $ms$ and dividing the resulting equation through by $m^n$, (3) becomes

$$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_{n-1}s+a_n=0,$$

which states, as desired, that $s$ is a root of (1).

COROLLARY. *To obtain an equation each of whose roots is equal numerically to a root of a given equation* (1), *but opposite in sign, change the signs of the coefficients of the terms of* **odd** *degree.*

Thus, the equation whose roots are equal numerically but opposite in sign to the roots of $2x^4+3x^3-x^2-4x+1=0$ is $2x^4-3x^3-x^2+4x+1=0$.

THEOREM II. *Having given an equation of the form*

$$(1) \qquad a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0,$$

*one may obtain an equation each of whose roots is less by a given amount $h$ than the corresponding root of* (1) *as follows. Divide* (1) *by $x-h$ and indicate the remainder by $R_n$, then divide the quotient by $x-h$ and indicate the remainder by $R_{n-1}$. Continue this process $n$ times, obtaining $a_0$ as the last quotient and $R_1$ as a last remainder. Then, the desired (transformed) equation is*

$$a_0x^n+R_1x^{n-1}+R_2x^{n-2}+\cdots+R_{n-1}x+R_n=0.$$

In applying this theorem, the various divisions should be performed by the method of synthetic division (§ 94).

Thus the process of finding the equation whose roots are each less by 2 than the roots of the equation $2x^3-19x^2+59x-60=0$ is, when arranged in condensed form, as follows.

$$\begin{array}{rrrrll}
2 & -19 & +59 & -60 & \underline{|2} & \\
 & 4 & -30 & +58 & & \\ \cline{1-4}
2 & -15 & +29 & \multicolumn{1}{|l}{-2} & & R_3=-2, \\
 & 4 & -22 & & & \\ \cline{1-3}
2 & -11 & \multicolumn{1}{|l}{+7} & & & R_2=+7, \\
 & +4 & & & & \\ \cline{1-2}
2 & -7 & & & a_0=2, & R_1=-7.
\end{array}$$

The coefficients of the desired new equation are therefore, in accordance with the above theorem, 2, −7, +7 and −2.

Hence, the required equation is $2x^3-7x^2+7x-2=0$.

Proof of Theorem 2. In order to obtain an equation whose roots are less by $h$ than the roots of (1), it suffices to replace $x$ throughout (1) by $x+h$, thus giving

$$a_0(x+h)^n+a_1(x+h)^{n-1}+\cdots+a_{n-1}(x+h)+a_n=0.$$

But, the various powers of $(x+h)$ here appearing may be expanded by the binomial theorem (§ 78) so that the last equation, after collection of terms and rearrangement according to descending powers of $x$, may be thrown into the form

$$(3)\qquad a_0x^n+A_1x^{n-1}+A_2x^{n-2}+\cdots+A_{n-1}x+A_n=0,$$

where $A_1, A_2, \cdots A_{n-1}, A_n$ are certain coefficients whose values we shall now determine.

From the manner in which we just obtained (3) from (1) it follows that, if we replace $x$ in (3) by $x-h$, we shall return to (1), that is, we may say that the following equation:

$$(4)\qquad a_0(x-h)^n+A_1(x-h)^{n-1}+A_2(x-h)^{n-2}+\cdots \\ +A_{n-1}(x-h)+A_n=0$$

is the same as (1). But the form of (4) shows that $A_n$ is equal to the remainder obtained by dividing the first member of (4) (or(1)) by $x-h$ that is, $A_n=R_n$. Similarly, $A_{n-1}$ is evidently the remainder obtained when the quotient of the last-named division is divided by $x-h$. Continuing this process to $n$ divisions, $A_1$ is the last remainder and $a_0$ the last quotient. Hence, in summary, we have, as required,

$$A_n=R_n,\ A_{n-1}=R_{n-1},\ \cdots\ A_1=R_1.$$

### EXERCISES

**1.** By use of Theorem 1, obtain the equation whose roots are 3 times the roots of the equation $3x^2-10x+3=0$, and verify the correctness of your result by solving both equations and examining the comparative sizes of their roots.

**2.** Obtain equations whose roots are equal to those of the following equations multiplied by the number opposite.

(*a*) $x^3-6x^2+x-1=0.$ (3)

(*b*) $x^4-3x^2+x+2=0.$ (−2)

(*c*) $x^3-\frac{x^2}{4}+\frac{x}{16}-\frac{1}{16}=0.$ (4)

(*d*) $2x^4-3x^2+5=0.$ (−3)

**3.** Obtain equations whose roots are equal to those of the following equations multiplied by the smallest number which will make all the coefficients integers and also make the coefficient of the highest power equal to unity.

(*a*) $3x^3-2x^2+x-1=0.$

[Hint. As the problem requires that the coefficient of the highest power of $x$ be 1, begin by dividing the equation through by 3, thus giving it the form $x^3-\frac{2}{3}x^2+\frac{1}{3}x-\frac{1}{3}=0$. Now write the equation whose roots are $m$ times the roots of this, and then assign to $m$ the *least* value necessary to make the new coefficients all integers.]

(*b*) $2x^4-5x^3+3x^2-2x-4=0.$

(*c*) $x^3-\frac{1}{2}x^2+\frac{1}{3}=0.$

(*d*) $3x^4+3x^2-5=0.$

(*e*) $2x^3-4x^2+1=0.$

**4.** Obtain the equations whose roots are numerically equal but of opposite sign to the roots of the equations in Exs. 2–3.

**5.** Obtain (using Theorem 2) equations whose roots are the roots of the following equations *diminished by* the number opposite.

(*a*) $x^3-12x^2+47x-60=0.$ (3)

(*b*) $2x^3-19x^2+59x-60=0.$ (2)

(*c*) $2x^4-3x^2+4x-5=0.$ (2)

(*d*) $2x^4-3x^2+4x-5=0.$ (−2)

(*e*) $x^4+9x^3+18=0.$ (−5)

(*f*) $x^5+3x+1=0.$ (1)

**97. Theorem Regarding Rational Roots.** Another general theorem which it is desirable to state before attempting to *solve* any equation of higher degree than the second (as we shall show how to do in §§ 98, 99) is as follows.

THEOREM. *An equation of the form*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0, \tag{1}$$

*wherein the coefficients $p_1, p_2, \cdots, p_n$ are all integers, can have no* ***rational*** *roots except integers (positive or negative).*

*Moreover, any integer that is a root will be an exact divisor of the last (constant) term, $p_n$.*

Thus, in the equation

$$x^3-x^2+2x+4=0,$$

the coefficient of the highest power of $x$ is 1, and all the remaining coefficients are integers. Hence, the only *possible* rational roots are the exact divisors of the last term, 4; namely 1, 2, 4, $-1$, $-2$, and $-4$. Whether any one (or more) of these *is* a root can be determined by the methods explained in § 95. It thus appears that none of the six values just mentioned is a root except $-1$. The fact that $-1$ *is* a root appears from the work below.

$$\begin{array}{r|l} 1-1+2+4 & \underline{-1} \\ \underline{\phantom{1}-1+2-4} & \\ 1-2+4+0 & \end{array}$$

NOTE. It will be recalled that *rational* numbers comprise all numbers of the form $a/b$, where $a$ and $b$ are integers (positive or negative). They therefore include such fractions as $\frac{1}{2}, \frac{2}{3}, \frac{6}{5}, -\frac{3}{4}$ etc., and all integers. This is in contrast to such numbers as $\sqrt{2}, \sqrt{3}, \sqrt[3]{2}$ etc., which cannot be so expressed, and are therefore called irrational. The roots of an equation may be all rational, all irrational, or partly one and partly the other. Also, some or all may be imaginary. Compare § 22.

PROOF OF THEOREM. Suppose that (1) had a rational root that was *not* an integer. Then this root could be expressed as a fraction in its lowest terms, $a/b$, where $a$ and $b$ are integers, and we would have

$$\left(\frac{a}{b}\right)^n+p_1\left(\frac{a}{b}\right)^{n-1}+p_2\left(\frac{a}{b}\right)^{n-2}+\cdots+p_{n-1}\left(\frac{a}{b}\right)+p_n=0. \tag{2}$$

Multiplying (2) through by $b^{n-1}$, we obtain

$$\frac{a^n}{b}+p_1a^{n-1}+p_2a^{n-2}b+\cdots+p_{n-1}ab^{n-2}+p_nb^{n-1}=0,$$

or

$$\frac{a^n}{b}=-(p_1a^{n-1}+p_2a^{n-2}b+\cdots+p_{n-1}ab^{n-2}+p_nb^{n-1}).$$

Since $a$ and $b$ as well as $p_1, p_2, \cdots, p_n$ are integers, the right member of the last equation likewise must be an integer. The left side, however, cannot be an integer since, if $a/b$ is a fraction in its lowest terms as we have supposed, it follows from arithmetic that $a^n/b$ will be again a fraction in its lowest terms.

Thus, we reach an absurdity upon the assumption that $a/b$ is a root. This leaves only integers as possible rational roots, as was to be shown.

To prove the last part of the theorem, suppose that $r$ is a root where $r$ is an integer. Then

$$r^n+p_1r^{n-1}+p_2r^{n-2}+\cdots+p_{n-1}r+p_n=0.$$

Transposing $p_n$ and dividing through by $r$, we obtain

$$r^{n-1}+p_1r^{n-2}+p_2r^{n-3}+\cdots+p_{n-1}=-\frac{p_n}{r}.$$

The left member of this equation is an integer since each term in it is an integer. Hence the quotient $p_n/r$ on the right must also be an integer, that is, $p_n$ must be exactly divisible by $r$, as was to be shown.

## EXERCISES

State all the *possible* rational roots of each of the following equations, and for each possibility determine, by the method of § 95, whether it *is* a root.

**1.** $x^3-4x^2-x+10=0.$

**2.** $x^3+5x^2-2x-10=0.$

**3.** $x^4-5x^3+4x^2-x+27=0.$

**4.** $x^4-15x^2-7x+12=0.$

**5.** $x^4+7x^3-x+18=0.$

**6.** $x^5-4x^3+x-2=0.$

## II. Determining the Real Roots of Any Equation

**98. Rational Roots.** We have seen how the theorem of § 97 affords a means of determining the rational roots of an equation *provided* the equation has the coefficient of its highest power of $x$ equal to 1 and the remaining coefficients are integers. We shall now illustrate how the rational roots of *any* equation may be obtained, provided only that the coefficients are rational numbers.

Example. Find the rational roots of the equation

$$3x^3+16x^2-3x-6=0. \tag{1}$$

Solution. Since the coefficient of the highest power of $x$ is *not* 1, the theorem of § 97 cannot be applied, hence we proceed as follows. First make the coefficient of $x^3$ equal to 1 by dividing through by 3:

$$x^3+\frac{16}{3}x^2-x-2=0.$$

Now transform this (by Theorem I, § 96) into an equation whose roots are 3 times as large:

$$x^3+3\cdot\frac{16}{3}x^2-3^2x-3^3\cdot 2=0,$$

or, reducing,

$$x^3+16x^2-9x-54=0. \tag{2}$$

The theorem of § 97 now applies to (2), indicating that its only possible rational roots are the integers ±1, ±2, ±3, ±6, ±9, ±18, ±27, ±54. Of these, the method of § 95 shows that +2 is the only value satisfying (2). The work for this case appears below.

$$\begin{array}{l|l} 1+16-\ 9-54 & \underline{2} \\ \underline{\ \ \ +\ 2+36+54} & \\ 1+18+27+0 & \end{array}$$

The corresponding depressed equation is seen to be

$$x^2+18x+27=0,$$

and, as the roots of this quadratic equation are at once found to be irrational (see § 22), it follows that the only rational root of (2) is 2.

Therefore, recalling that each root of (2) is three times the corresponding root of (1), it follows that (1) has but one rational root, the value of which is one-third of 2, or 2/3.

Similarly, the rational roots of any equation

$$f(x)=0$$

whose coefficients are themselves rational numbers may be found by the following rule:

RULE FOR DETERMINING RATIONAL ROOTS. *Divide both members of the equation by the coefficient of the highest power of $x$, thus obtaining* 1 *as its new coefficient.*

*Transform this equation into one whose roots are $m$ times as large, choosing $m$ in such a way that the coefficients of the new equation will all be integers.*

*Determine the integral solutions of the last equation by trial, using the theorem of* § 97, *and divide each root thus obtained by $m$.*

## EXERCISES

Find the rational roots and if possible all the roots of each of the following equations.

**1.** $3x^3+2x^2-4x+1=0.$

**2.** $2x^3-x^2-7x+6=0.$

**3.** $2x^4-3x^3-20x^2+27x+18=0.$

**4.** $2x^4-9x^3-27x^2+134x-120=0.$

**5.** $24x^3-34x^2-5x+3=0.$

**6.** $18x^3+3x^2-7x-2=0.$

**7.** $9x^3-27x^2+23x-5=0.$

**8.** $2x^3-11x^2+8x+7=0.$

**9.** $72x^4+90x^3-5x^2-40x-12=0.$

**99. Irrational Roots. Horner's Method.** Suppose that the given equation is

$$(1) \qquad f(x)=x^3+3x^2-10x-6=0.$$

In this case the only possible rational roots, as indicated by § 97, are ±1, ±2, ±3 and ±6, but none of these, when tested as in § 95, satisfies the equation. Hence, any real roots that can be present must be *irrational.* If such roots are to be determined correct to any given place of decimals, it is best to begin by sketching the graph of the given function, $x^3+3x^2-10x-6$, thus obtaining an *approximate* value for each of the roots by inspection, as in § 86.

The graph may be drawn readily as follows. If we place $y=x^3+3x^2-10x-6$, the value of $y$ when $x=3$, for example, will be the remainder obtained by dividing $x^3+3x^2-10x-6$ by $x-3$ (see § 93). This remainder may be calculated rapidly by synthetic division, as below.

$$\begin{array}{r|l} 1+3-10-\ 6 & \underline{3} \\ +3+18+24 & \\ \hline 1+6+\ 8+\mathbf{18} & \end{array}$$

Hence, when $x=3$, $y=+18$. Similarly, the value of $y$ corresponding to any given value of $x$ may be found. The graph is as indicated in Fig. 54, where, for the convenience of the drawing, each space along the $y$-axis is counted as 5 units. Three real roots are thus seen to be present. In particular, one root lies between 2 and 3 and we shall now proceed to determine with accuracy this particular root, following the process known as *Horner's Method.* The other two roots could be determined similarly if desired, as will be shown later.

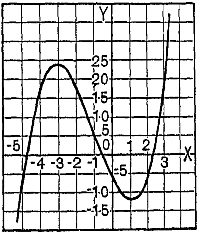

FIG. 54

Since the root in question lies between 2 and 3, we first transform the equation into one whose roots are each less by 2 than those of the original equation, using for this purpose Theorem II of § 96. The work appears below.

$$\begin{array}{llll}
1+3 & -10 & -6 & \underline{|2} \\
\phantom{1}+2 & +10 & +0 & \\ \hline
1+5 & +\ 0 & \multicolumn{1}{|l}{\mathbf{-6}} & \\
\phantom{1}+2 & +14 & & \\ \hline
1+7 & \multicolumn{1}{|l}{\mathbf{+14}} & & \\
\phantom{1}+2 & & & \\ \hline
\mathbf{1+9} & & &
\end{array}$$

Hence the transformed equation is

$$x^3+9x^2+14x-6=0. \tag{2}$$

Recalling what has been said of the roots of (1) and that the roots of (2) are each less by 2 than those of (1), we see that the root of (2) in which we are interested lies between 0 and 1. Equation (2) may be called the *first* transformed equation.

We proceed now to note the changes in value which the first member of (2) undergoes as $x$ ranges by successive *tenths* from 0 to 1, that is, we evaluate this member, using the abridged method already explained, when $x$ takes successively the values 0.0, 0.1, 0.2, 0.3, ⋯, 0.9. It is thus found (in particular) that when $x=0.3$ this member equals $-0.963$; while if $x=0.4$, it equals $+1.104$. The work is shown below.

$$\begin{array}{l|l}
1+9.0+14.00-6.000 & \underline{0.3} \\
\phantom{1}+0.3+\ \ 2.79+5.037 & \\ \hline
1+9.3+16.79\mathbf{-0.963} &
\end{array}
\qquad
\begin{array}{l|l}
1+9.0+14.00-6.000 & \underline{0.4} \\
\phantom{1}+0.4+\ \ 3.76+7.104 & \\ \hline
1+9.4+17.76\mathbf{+1.104} &
\end{array}$$

Noting from this that when $x=0.3$ the first member of (2) is *negative* in value, while for $x=0.4$ it is *positive*, we see that this member, when regarded as a function of $x$, must be *equal* to zero for *some* value of $x$ lying between 0.3 and 0.4 In other words, (2) must have a root between these two values.

Recalling that the roots of (1) are 2 greater than those of (2), we see, in turn, that (1) must have a root between 2.3 and 2.4, so that the root of (1) in which we are interested, when computed correct to *one* place of decimals, is 2.3 We proceed now to get this root correct to *two* places of decimals, and finally to *three*, the process admitting of indefinite continuation, so that the root in question may be determined as accurately as one desires. Less labor, is required to determine the digits of the decimal beyond the one in tenth's place.

Transforming (2) into an equation whose roots are .3 less,

$$\begin{array}{rrrr|l}
1 & +9.0 & +14.00 & -6.000 & \underline{.3} \\
 & 0.3 & +\ 2.79 & +5.037 & \\
\hline
1 & +9.3 & \multicolumn{1}{r|}{+16.79} & \mathbf{-0.963} & \\
 & +0.3 & \multicolumn{1}{r|}{+\ 2.88} & & \\
\hline
1 & \multicolumn{1}{r|}{+9.6} & \mathbf{+19.67} & & \\
 & \multicolumn{1}{r|}{+0.3} & & & \\
\hline
\mathbf{1} & \mathbf{+9.9} & & &
\end{array}$$

we find the *second* transformed equation to be

(3) $$x^3+9.9x^2+19.67x-0.963=0.$$

Since the root, $x$, of (2) in which we are interested lies between .3 and .4, and each root of (3) is .3 less than the corresponding root of (2), the root of (3) which we are to determine lies between 0 and .1 Hence it is relatively small. In fact, it is so small that we may, with reasonable safety, drop off the terms of (3) which contain higher powers of $x$ than the first, since they are very small in comparison to $x$ itself. The equation then reduces (3) to the simple form

(4) $$19.67x-0.963=0,$$

whose solution is evidently $0.963 \div 19.67$, or, approximately, $.04^+$ Hence, although the root of (3) which we are seeking is not *exactly* equal to the solution of (4), its value, when computed merely to the first *significant* figure, may safely be taken as .04

**Note 1.** In order to remove all existing doubt at this point, one may determine (by the usual synthetic process) the values of the first member of (3) when $x=.04$ and $x=.05$ respectively. If the results are of *opposite* sign, no mistake has been made in taking .04 as the root desired (correct to the first significant figure) of (3), but if the results are of the *same* sign, the root can evidently not lie between .04 and .05 One should in such cases proceed to find also the results for .03 and .06 to ascertain between what two consecutive hundredths the change of sign in the left member of (3) *does* occur. It is usually desirable to check in this way the value which has been determined as a *probable* value of the root, especially if it is greater than .05, but it is usually *not* necessary to check the similar tentative roots obtained from time to time in continuing the process which follows below.

It follows that the root of (2) in which we are interested, correct to *two* decimal places, is .34; hence the desired root of (1) to a similar degree of accuracy is $x=2.34$

In order to determine the next figure of the root, we now proceed as before, that is, we first transform (3) into an equation whose roots are less by .04 The work appears below.

$$\begin{array}{rrrr|l}
1+ & 9.90 & +19.6700 & -0.963000 & \underline{.04} \\
+ & .04 & +\ \ .3976 & +0.802704 & \\
\hline
1+ & 9.94 & +20.0676 & \mathbf{-0.160296} & \\
+ & .04 & +\ \ .3992 & & \\
\hline
1+ & 9.98 & \mathbf{+20.4668} & & \\
+ & .04 & & & \\
\hline
\mathbf{1+} & \mathbf{10.02} & & &
\end{array}$$

Thus the *third* transformed equation is therefore

$$x^3+10.02x^2+20.4668x-0.160296=0, \tag{5}$$

and its root in which we are interested must lie between 0 and .01 To obtain it to the first significant figure, we solve the equation

$$20.4668x-0.160296=0 \tag{6}$$

thus obtaining $x=.007^+$ Hence the root of (1), correct to three decimal places, is $x=2.347$ Evidently the process may now be continued indefinitely, thus determining the root in question to any desired degree of accuracy. It is to be noted finally that the preceding work may be conveniently and compactly arranged as follows.

$$\begin{array}{llll|l}
1+3 & -10 & -6 & & \underline{2} \\
\phantom{1}+2 & +10 & +0 & & \\
\hline
1+5 & +\ 0 & \multicolumn{1}{|l}{-6} & & \\
\phantom{1}+2 & +14 & \multicolumn{1}{|l}{} & & \\
\hline
1+7 & \multicolumn{1}{|l}{+14} & & & \\
\phantom{1}+2 & \multicolumn{1}{|l}{} & & & \\
\hline
\mathbf{1+9} & \mathbf{+14} & \mathbf{-6} & & \underline{.3} \\
\phantom{1}+0.3 & +\ 2.79 & +5.037 & & \\
\hline
1+9.3 & +16.79 & \multicolumn{1}{|l}{-0.963} & & \\
\phantom{1}+0.3 & +\ 2.88 & \multicolumn{1}{|l}{} & & \\
\hline
1+9.6 & \multicolumn{1}{|l}{+19.67} & & & \\
\phantom{1}+0.3 & \multicolumn{1}{|l}{} & & & \\
\hline
\mathbf{1+9.9} & \mathbf{+19.67} & \mathbf{-0.963} & & \underline{.04} \\
\phantom{1}+\ .04 & +\ \ .3976 & +0.802704 & & \\
\hline
1+9.94 & +20.0676 & \multicolumn{1}{|l}{-0.160296} & & \\
\phantom{1}+\ .04 & +\ \ .3992 & \multicolumn{1}{|l}{} & & \\
\hline
1+9.98 & \multicolumn{1}{|l}{+20.4668} & & & \\
\phantom{1}+\ .04 & \multicolumn{1}{|l}{} & & & \\
\hline
\mathbf{1+10.02} & \mathbf{+20.4668} & \mathbf{-0.160296} & & \underline{.007}
\end{array}$$

In summary, then, we have the following rule.

Rule for Determining a Positive Irrational Root.

1. *Sketch the graph and thus locate the root between two consecutive integers (subject to the remarks in Note 2 below).*

2. *Obtain an equation whose roots are less than those of the given equation by the smaller of these two integers. This equation will have the root in question lying between* 0 *and* 1.

3. *Locate this root (by trial) between two successive tenths, and obtain a new equation whose roots are less than those of the last one by the smaller of these tenths. This equation will have the root in question lying between* 0 *and* 0.1

4. *Locate this root correct to its first significant figure by Horner's Method of approximation (subject to the remarks in Note* 1 *above) and obtain a new equation whose roots are less than those of the last one by the smaller of the hundredths thus determined. This equation will have the root in question lying between* 0 *and* 0.01

5. *Continue the process to any required number of decimal places.*

6. *The sum of all the diminutions of the roots gives the value of the required root correct to the last decimal place appearing in the process.*

In order to determine a *negative* irrational root of an equation $f(x)=0$, we have merely to determine the corresponding positive root of the equation $f(-x)=0$. See corollary, § 96.

NOTE 2. It may happen that two (or more) roots of a given equation are so nearly equal that it is difficult to distinguish between them on the graph and hence difficult to obtain for each a first approximation that will be different in the two cases. Under such circumstances, it is necessary to begin by determining each *by trial* correct to the first place of decimals rather than merely to the first integer. For example, the equation

$$f(x)=4x^3-24x^2+44x-23=0$$

has *two* roots lying between 2 and 3, as appears upon sketching its graph, which is shown in Fig. 55. By evaluating $f(x)$ as $x$ takes on the successive values 2, 2.1, 2.2, 2.3, ⋯, 2.9, 3, we see that $f(x)$ changes sign between $x=2.2$ and $x=2.3$, and again between 2.8 and 2.9 One root, correct to *one* decimal place, is therefore 2.3, and another is 2.8 Either may now be determined accurately by the transformation process described above, combined with Horner's Method.

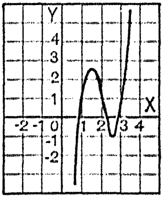

FIG. 55

The cases in which two or more roots are actually equal can be treated by introducing the notion of the derivative (§ 87) but the detailed explanation of the method will not be attempted here. In such cases the $x$-axis is a *tangent* line to the graph. When two roots thus coincide they are said to form a *double root,* when three roots coincide they form a *triple root,* etc.

## EXERCISES

Determine each of the following roots correct to three decimal places, accompanying each equation with its graph.

**1.** The root of $x^3-3x^2+6x-9=0$ lying between 2 and 3.

**2.** The root of $x^3-3x^2-3x-7=0$ lying between 4 and 5.

**3.** The root of $x^3+13x^2+57x-16=0$ lying between 0 and 1.

**4.** The root of $x^3+6x^2+9x+1=0$ lying between $-3$ and $-4$.

**5.** The root of $x^4+4x^3-4x^2-12x+3=0$ that lies between 1 and 2.

**6.** Determine, correct to *two* decimal places, the roots of the equation $x^4-6x^3+5x^2+14x-4=0$ between 3 and 4. See Note 2, § 99.

**7.** Determine, by use of Horner's Method, the value of the fourth root of 473 correct to four places of decimals. Note that this is equivalent to solving the equation $x^4=473$.

**8.** Determine, correct to three decimal places, the fifth root of 100.

**9.** Obtain, correct to two decimal places, the positive solutions $x$, $y$ of the following simultaneous quadratic equations (compare §§ 29, 30) $xy=1$, $y=x^2-2$.

**10.** The edge of a cube measures 3 inches. By how much, correct to two decimal places, should each edge be increased in order that the volume may be increased by 50 cubic inches?

**11.** The dimensions of a rectangular box are 8 by 10 by 12 inches. By what amount, correct to three decimal places, should each be increased in order that the volume may be increased by 400 cubic inches?

**12.** How long is the edge of a cube if, after cutting off a slice 3 inches thick from one side, there remain 20 cubic inches?

**13.** A right circular cylinder has its upper base hollowed out into the form of a hemisphere. In order that the solid thus formed may have the same volume as a sphere 4 feet in diameter, determine, correct to three decimal places, what must be the radius of its base if the height is 10 feet.

[HINT. See formulas in § 7.]

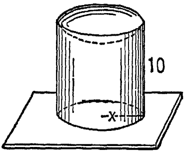

FIG. 56

**14.** Answer Ex. 13 in case both bases of the cylinder are hollowed out in the manner indicated.

**15.** The depth of flotation in water of a material sphere is the positive root of the equation $x^3-3rx^2+4r^3s=0$, where $r$ is the radius and $s$ is the specific gravity of the material. Find, correct to two decimal places, the depth at which a cork sphere of radius 1 foot will sink, it being given that the specific gravity of cork is 0.24

**100. Algebraic Solutions.** It has been shown in an earlier chapter that if one considers the general quadratic equation, namely

$$ax^2+bx+c=0, \tag{1}$$

it is possible to determine formulas for its two roots, thus expressing them in all cases in terms of the coefficients $a$, $b$, $c$. In fact, it was shown in § 21 that the roots of (1) are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{2}$$

Similarly, one may now inquire whether formulas exist which express the three roots of the general cubic, namely, the three roots of the equation

$$ax^3+bx^2+cx+d=0. \tag{3}$$

Such formulas exist, but they are difficult to use, and are of theoretic interest only. We shall therefore merely state the following facts. Equation (3) may be transformed into the more simple form

$$x^3+gx+h=0 \tag{4}$$

and the roots of (4) are given by the formula

$$\sqrt[3]{-\frac{h}{2}+\sqrt{\frac{g^3}{27}+\frac{h^2}{4}}}+\sqrt[3]{-\frac{h}{2}-\sqrt{\frac{g^3}{27}+\frac{h^2}{4}}}.$$

Since any given quantity has *three* cube roots (real or imaginary), this formula determines the three roots of (4), just as (2) determines the two roots of (1).

Similarly, the general equation of the fourth degree (commonly called the general *biquadratic*, or the general *quartic*) may be solved, the formulas for its roots being, however, highly complicated. As regards the general equation of the *fifth* degree (*quintic*) and all higher degrees, it is *not* possible in general to express their solutions in terms of radicals.

# CHAPTER XII

## PERMUTATIONS AND COMBINATIONS

**101. Introduction.** Consider the following question. How many signals may be given by hoisting 2 flags on a pole, it being understood that there are 10 flags of different colors to select from?

The answer can be reasoned out as follows. The first flag may be chosen in any one of 10 ways and, having chosen it, the second flag may be chosen in any one of 9 ways. Since to *each* of the 10 choices of the first flag there thus correspond 9 choices of the second, the answer must be $10 \times 9$, or 90 signals.

Again, if we ask in how many ways 3 letters may be mailed on a street where there are 5 letter-boxes, we may reason as follows: The first letter may be mailed in any one of 5 ways, and, having been mailed, the second letter may likewise be mailed in any one of 5 ways. Hence, as in the example above, the first two letters may be mailed in $5 \times 5$, or 25 ways. But to *each* of these correspond 5 ways also of mailing the third letter, hence the number of ways in which all three letters may be mailed is $25 \times 5$, or 125 ways.

If a man can travel on any one of four routes from New York to Buffalo, and thence on any one of three routes from Buffalo to Chicago, he may make the whole trip (via Buffalo) in any one of twelve routes.

Similar reasoning leads to the following general principle.

FUNDAMENTAL PRINCIPLE. *If one thing can be done in* $m_1$ *different ways, and, having done it, a second thing can be done in* $m_2$ *different ways, and having done it, a third thing can be done in* $m_3$ *different ways, and so on, then the number of ways in which the various things can be done jointly is the product* $m_1 \cdot m_2 \cdot m_3 \cdots$.

## EXERCISES

**1.** How many signals can be given by hoisting 3 flags if there are 8 different flags to select from?

**2.** In how many ways can 4 letters be mailed if there are 5 mail boxes?

**3.** In how many ways can 4 different positions be filled if there are 3 applicants for the first position, 2 for the second and 4 for each of the others?

**4.** Answer Ex. 3 in case there are 12 applicants in all, each of whom is eligible to either place.

**5.** If a person owns a 5-seated automobile, in how many ways can he seat a party of four for a ride?

**6.** How many base-ball nines can be formed out of 9 men, it being understood that any man can play in any place?

**7.** Answer Ex. 6 in case either A or B must pitch, while either B or C must catch.

[HINT. Solve first on the supposition that A pitches and B catches. Then consider similarly all possibilities and *add* results.]

**8.** How many signals can be given with 6 different colored flags which may be hoisted either singly or any number at a time?

**9.** How many *even* numbers can be formed using the digits 1, 2, 3, 4, 5, 6, 7, it being understood that all of the digits are to be used and each used but once?

[HINT. Determine first how many ways the *last* digit of the number may be chosen.]

**10.** Answer Ex. 9 in case any number of the digits may be used, but no digit more than once.

**11.** In how many ways can an ace, king, queen and jack be drawn from a pack of cards in the order named in case

(*a*) they may be of different suits,
(*b*) they must be of different suits,
(*c*) they must be of the same suit?

**12.** If a half-dollar, quarter-dollar, dime and nickel be tossed, in how many ways can they come up?

**13.** If there are four convenient routes from Chicago to San Francisco, in how many ways can one conveniently make the round-trip?

**14.** In how many ways can one draw a square 1 inch on a side if he has black, red and green ink at his disposal, using only one color on any one side?

**102. Permutations.** Consider the three letters $a$, $b$, $c$, and let it be asked how many different arrangements, or ***permutations,*** of these letters among themselves are possible. The answer is six, as all such arrangements, or permutations, are evidently the following:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba.$$

We might have asked a different question as follows. How many permutations are possible with the four letters $a$, $b$, $c$, $d$ in case only *two* of them are used at a time. The answer would now be twelve, such permutations being

$$ab,\ ba,\ ac,\ ca,\ ad,\ da,\ bc,\ cb,\ bd,\ db,\ cd,\ dc.$$

In general, if we have $n$ objects (regarded as different from each other) there will be a certain number of possible arrangements, or permutations, of them when taken $r$ at a time. If we represent the number of such permutations, as is customary, by the symbol ${}_nP_r$, we may show that ${}_nP_r$ is determined by the formula

$$(1) \qquad {}_nP_r = n(n-1)(n-2) \cdots (n-r+1).$$

For, the object to be placed *first* in the arrangement may evidently be chosen in any one of $n$ ways, and, having chosen it, but $n-1$ objects remain, so that the object to be placed *second* may be chosen in any one of $n-1$ ways; similarly, the *third* object may be chosen in any one of $n-2$ ways, the *fourth* object in any one of $n-3$ ways, and so on, until finally the last, or $r$th object may be chosen in any one of $n-r+1$ ways. Hence, applying the fundamental principle stated in § 101, we see that the total number of ways of arranging, or permuting, the $n$ objects when taken $r$ at a time will be the product $n(n-1)(n-2) \cdots (n-r+1)$. That is, we arrive at formula (1).

Thus, the number of possible permutations of the 4 letters $a$, $b$, $c$, $d$ when taken 3 at a time is, in accordance with (1), equal to $4 \cdot 3 \cdot 2$, or 24.

This may be verified by actually writing out all such permutations, the result being as shown below.

| | | | |
|---|---|---|---|
| *abc* | *bac* | *cab* | *dab* |
| *acb* | *bca* | *cba* | *dba* |
| *acd* | *bcd* | *cbd* | *dbc* |
| *adc* | *bdc* | *cdb* | *dcb* |
| *abd* | *bad* | *cad* | *dac* |
| *adb* | *bda* | *cda* | *dca* |

Similarly, the number of permutations of 5 letters when taken *all* at a time would be determined by formula (1) by placing in it $n=5$ and $r=5$, the result thus being $5\cdot 4\cdot 3\cdot 2\cdot 1$, or 120. If, on the other hand, we use only 3 of the letters at a time, the result would be $5\cdot 4\cdot 3$, or 60. If we use 2 at a time, the result would be $5\cdot 4$ or 20, etc.

**103. The Factorial Numbers.** If in formula (1) we place $r=n$, the last factor becomes $n-n+1$, or 1, so that the right member becomes

$$n(n-1)(n-2)\cdots 2\cdot 1.$$

This expression, which represents the product of all the integers from 1 to $n$ inclusive, is called ***factorial n***, and is commonly designated by the symbol $n!$

Thus, $3!=1\cdot 2\cdot 3=6$; $4!=1\cdot 2\cdot 3\cdot 4=24$; $5!=1\cdot 2\cdot 3\cdot 4\cdot 5=120$, $6!=720$, $7!=5040$, $8!=40320$, $9!=362880$, $10!=3628800$, etc.

NOTE. From the definition of $n!$ it follows that, whatever the value of $n$, we shall have $n!=n\cdot(n-1)!$. Placing $n=1$ in this relation gives $1!=1\cdot 0!$, or $1=0!$. Hence, the value of $0!$ must be taken as 1. (Compare the meaning of $a^0$ as obtained in § 8).

Inasmuch as $n!$ is the result of placing $r=n$ in formula (1), it follows that *the number of permutations of $n$ things taken all at a time is $n!$*

Expressed as a formula, this result becomes

$$(2)\qquad {}_nP_n=n(n-1)\cdots 2\cdot 1=n!$$

Thus, the five letters $a$, $b$, $c$, $d$, $e$ may be permuted among themselves in $5!=120$ ways.

### EXERCISES

**1.** In how many ways can the letters *a*, *b*, *c*, *d*, *e* be arranged if taken 3 at a time?

**2.** How many numbers can be made out of the digits 1, 2, 3, 4, 5, 6 using four of them at a time, no digit being repeated?

**3.** In how many ways can 10 trees be planted in a row?

**4.** In how many ways can the letters *A*, *B*, *C*, *a*, *b*, *c* be arranged so that the three capital letters shall stand first, and the three small letters shall stand last?

[HINT. First find how many ways the capital letters can be arranged among themselves, then similarly as regards the small letters, then use the Principle of § 101.]

**5.** Work Ex. 4 in case either the three capital letters or the three small letters may stand first.

**6.** In how many ways can 5 French books, 3 Latin books and 2 Spanish books be arranged on a shelf so that the French books shall stand together, the Latin books together, and the Spanish books together?

**7.** Work Ex. 6 when it is required that the French books shall stand first as a group, but the remaining 5 books may be arranged in any manner thereafter.

**8.** In how many ways can a program of 3 speeches and 3 musical numbers be arranged so that speeches and music shall alternate throughout?

**9.** In how many ways can the knives, forks and spoons be distributed at a table where there is to be a dinner party of 6 people, each of whom is to have a dinner knife, a bread and butter knife, a dinner fork, a salad fork, a soup spoon, a teaspoon and a coffee spoon?

**10.** In how many ways can the colors red, green, blue, indigo, violet be arranged so that red and green do *not* stand together?

[HINT. The answer may be regarded as the difference between the number of arrangements when no restrictions whatever are made and the number when red and green stand together in either order.]

**11.** In how many ways can 4 different coins be stacked one upon the other provided that at least one must be left with its face side up?

**12.** Show that formula (1) of § 102 may be written in the form

$$_nP_r = \frac{n!}{(n-r)!}.$$

**104. Combinations.** A set of things regarded without reference to the order in which they are arranged, is called a *combination* of them.

Thus, *abc*, *acb*, *bac*, *bca*, *cab* and *cba* are the same combination because each is made out of the same letters $a$, $b$, $c$ and in this respect there is no difference between them. It is only when the arrangement of the letters is taken into account that any such distinctions are possible.

Let us ask how many combinations, in the sense defined above, are possible out of the four letters $a$, $b$, $c$, $d$ when taken 3 at a time. The answer is 4; namely, *abc*, *abd*, *acd*, *bcd*. Note that each of these is different from the three others in that it is made up of different letters. Similarly, if we ask how many combinations of the letters $a$, $b$, $c$, $d$ are possible when taken 2 at a time, the answer is 6; namely *ab*, *ac*, *ad*, *bc*, *bd*, *cd*. Finally, if taken 4 at a time, the answer is 1; namely *abcd*.

If we ask in a more general sense how many combinations are possible out of $n$ different things taken $r$ at a time, we may arrive at a formula for it as follows. Consider any one combination. It contains $r$ letters, which, if arranged in all possible ways would give rise to $r!$ permutations. (See formula (2), § 103.) Since this is true of every different combination, it follows that if we let ${}_nC_r$ represent the total number of such combinations, we shall have the equation

$${}_nC_r \cdot r! = {}_nP_r,$$

where ${}_nP_r$ is the total number of permutations of the $n$ things taken $r$ at a time. From this equation we have

$${}_nC_r = \frac{{}_nP_r}{r!},$$

which, when we substitute for ${}_nP_r$ its value as given by (1), § 102, becomes

$$(1) \qquad {}_nC_r = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

This, therefore, is the formula desired. By multiplying both its numerator and denominator by $(n-r)!$, observing that the numerator then becomes

$$n(n-1)\cdots(n-r+1)\cdot(n-r)! =$$
$$n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 1 = n!,$$

the formula takes the more condensed form

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!}. \tag{2}$$

**NOTE.** It may be noted that formula (1) for ${}_nC_r$ is the same as is obtained if, in the formula as given in § 79 for the coefficient of the $r$th term of the binomial expansion for $(a+x)^n$, one uses $(r+1)$ in place of $r$. The binomial theorem for positive integral exponents may therefore be written in the form

$$(a+x)^n = a^n + {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 + \cdots + {}_nC_{n-1}ax^{n-1} + {}_nC_nx^n.$$

EXAMPLE 1. How many committees of 3 men each can be formed from 8 men?

SOLUTION. Since the personnel of a committee is in nowise changed by a different arrangement of the men in it, the question resolves itself into finding the number of *combinations* of 8 men when taken 3 at a time.

Hence, using the first of the formulas above, we obtain the answer

$${}_8C_3 = \frac{8\cdot 7\cdot \cancel{6}}{1\cdot \cancel{2}\cdot \cancel{3}} = 56.$$

EXAMPLE 2. How many selections each consisting of 3 oranges and 2 apples may be made from a basket containing 6 oranges and 4 apples?

SOLUTION. The number of ways in which the 3 oranges may be selected is

$${}_6C_3 = \frac{\cancel{6}\cdot 5\cdot 4}{1\cdot \cancel{2}\cdot \cancel{3}} = 20.$$

The number of ways in which the 2 apples may be selected is

$${}_4C_2 = \frac{\overset{2}{\cancel{4}}\cdot 3}{1\cdot \cancel{2}} = 6.$$

Hence, by the fundamental principle of § 101, the 3 oranges and 2 apples may together be selected in 20×6 = 120 ways. *Ans.*

### EXERCISES

**1.** A captain having under his command 20 men wishes to form a guard of 3 men. In how many ways may the guard be formed?

**2.** How many peals may be rung with 7 different bells by striking them 4 at a time?

**3.** How many hands of cards, each made up of 5 hearts, are there in a pack of cards?

[HINT. The pack contains 52 cards, there being 13 each of hearts, diamonds, spades and clubs.]

**4.** A chandelier contains 10 lights. In how many ways may the room be lighted if only 8 lights are used?

**5.** How many straight lines may be drawn through 8 points no three of which lie in the same straight line?

**6.** Out of 8 different English books and 7 different French books, how many selections of 6 books may be made each containing 3 English and 3 French books?

**7.** Work Ex. 6 in case each selection of 6 books must contain *at least* 2 English and 2 French books?

[HINT. Consider separately the various possibilities, as in Ex. 7, page 183, and *add* results.]

**8.** A candidate for a certain office is to be elected in case he receives a majority of the votes cast by 10 people. In how many ways could the majority be secured?

**9.** Out of 15 men how many selections of 4 men each can be made each of which will contain a certain particular man?

[HINT. Take out the particular man and then consider the remaining 14 men.]

**10.** A whist-hand contains 13 cards. How many such hands each made up of 4 spades, 4 hearts, 4 diamonds and 1 club is it possible to form?

**11.** Out of a basket containing 6 oranges, 8 apples and 3 peaches, how many selections of 5 each may be made that shall contain at least one orange?

[HINT. The answer may be regarded as the difference between the number of selections of 5 indiscriminately and the number when no oranges are taken.]

**12.** Show that the number of combinations of $n$ things taken $r$ at a time is the same as when taken $n-r$ at a time.

***105. Distribution into Groups.** If it be asked in how many ways 10 different things may be distributed among 3 persons A, B, C so that A shall receive 5, B shall receive 3, and C shall receive 2, the answer may be determined as follows. Starting with A, he may receive his 5 things in

$${}_{10}C_5 = \frac{10!}{5!5!} \text{ ways.} \qquad \text{(See formula (2), § 104)}$$

B may now be given his 3 things out of the remaining 5 things in

$${}_5C_3 = \frac{5!}{3!\,2!} \text{ ways.}$$

Finally, C may be given his 2 things out of the remaining 2 things in

$${}_2C_2 = \frac{2!}{2!\,0!} = \frac{2!}{2!} \text{ ways.} \qquad \text{(See Note, § 103)}$$

Applying the Theorem of § 101, the 10 things may therefore be distributed in the manner specified in

$$\frac{10!}{5!\,5!} \times \frac{5!}{3!\,2!} \times \frac{2!}{2!} \text{ ways.}$$

Noting cancellations, we may reduce this product to the form

$$\frac{10!}{5!\,3!\,2!} = \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot 7 \cdot \overset{4}{\cancel{8}} \cdot 9 \cdot 10}{\cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{2}} = 2520 \text{ ways. } \textit{Ans.}$$

The same method of reasoning when applied more generally leads to the following result.

*The number of ways in which n different things may be distributed into a specified number of groups such that the first group shall contain p things, the second shall contain q things, the third shall contain r things, etc. is given by the formula*

$$N = \frac{n!}{p!\,q!\,r!\cdots}. \tag{1}$$

Example. In how many ways may 14 apples be distributed among four children so that the oldest shall receive 5, the next younger 4, the next younger 3 and the youngest 2.

Solution. By means of the above general formula, the answer is

$$\frac{14!}{5!\,4!\,3!\,2!} = 2{,}522{,}520 \text{ ways. } \textit{Ans.}$$

**Note.** It is to be observed that if the number of things to be put in each group is the *same* that is, $p=q=r=\cdots$, and if there is no distinction made between the groups (such as first, second, etc.), then the formula above must be slightly changed, becoming

$$N=\frac{n!}{g!\,p!\,q!\,r!\cdots}. \tag{2}$$

where $g$ is the number of the groups. The reason for this may be immediately implied from the following example.

Example. In how many ways may 12 men be divided into three groups of 4 each?

Solution. Formula (1) would give

$$\frac{12!}{4!\,4!\,4!}.$$

But to take this as the answer implies that any way of dividing the men into the three equal groups gives rise to another way by redistributing the same three groups among themselves, which can be done in 3! ways. Since the question is merely as to the number of possible groups without reference to their order, the result above must therefore be divided by 3!, giving as the correct answer

$$\frac{12!}{3!\,4!\,4!\,4!}=5775 \text{ ways}$$

and thus agreeing with the result given by (2) for this example.

***106. Permutations of Things not all Different.** In the previous discussions of this chapter all the things dealt with have been regarded as different, or distinguishable, from each other. In distinction from this, consider the following example.

Example. How many permutations are possible of the letters of the word *infinite* when taken all together?

Solution. Since no new permutation arises by interchanging the three $i$'s among themselves, or the two $n$'s among themselves, let us suppose at first that the $i$'s are made dissimilar by calling them respectively $i_1$, $i_2$, $i_3$, and likewise the $n$'s by calling them respectively $n_1$, $n_2$. Under such a supposition, the answer, by formula (2) of § 103, would be 8!, since there would then be a total of 8 dissimilar letters. If the three $i$'s be now regarded as the same, *each* of these 8! permutations gives rise (by permuting the 3 $i$'s among themselves) to 3! permutations that are identically the same. Hence, if the $i$'s alone be regarded as the same, the answer would be 8!/3!. But if the two $n$'s be now regarded as the same, *each* of these 8!/3! permutations gives rise by similar reasoning to 2! permutations that are the same. Hence, the correct answer is

$$\frac{8!}{3!\,2!}=3360.$$

The same method of reasoning when applied more generally leads to the following result:

*The number of permutations among themselves of $n$ things of which $n_1$ are alike of one kind, $n_2$ are alike of another kind, $n_3$ are alike of another kind, etc., is given by the formula*

$$P=\frac{n!}{n_1!\ n_2!\ n_3!\ \cdots}.$$

## MISCELLANEOUS EXERCISES

Success in working an example in permutations and combinations depends chiefly upon one's ability to determine to what extent the *order* of the things considered must be taken into account. Examples in the following list accompanied by the star (*) depend upon §§ 105–106.

**1.** On a railroad there are 20 stations. How many tickets are required to connect every station with every other one?

**2.** The Greek alphabet contains 24 letters. How many Greek letter fraternity names can be formed, each containing 3 letters, a repetition of letters being allowed?

**3.** In how many ways can 6 ladies and 6 gentlemen form couples for a dance?

***4.** Eight persons are to play cards. In how many ways can partners be formed?

**5.** Show that the number of ways in which $n$ persons may be distributed among themselves at a round table is $(n-1)!$

**6.** In how many ways can a selection of at least 4 oranges be made from a basket of 8 oranges?

**7.** A box contains 6 red cards, 5 white cards and 4 blue cards. In how many ways can a selection of three cards be made such that

(*a*) all three are red?
(*b*) none are red?
(*c*) at least one is red?

***8.** How many arrangements of the letters of the word Mississippi are possible?

***9.** How many signals can be made with 7 flags, of which 2 are red, 1 white, 3 blue and 1 yellow, displayed altogether one above the other?

**10.** How many dominoes are there in a set numbered from double blank to double ten?

***11.** A collection of 12 books is to be distributed equally among 4 people. In how many ways can it be done, no regard being had for the *order* in which they are given out?

***12.** A collection of 12 books is to be divided into 4 equal piles. In how many ways can it be done, no regard being had for the *order* in which they appear in each pile?

**13.** Answer Ex. 12 in case regard is taken of the order of the books in each pile.

**14.** How many committees, each containing 4 men, can be formed from 5 Republicans and 5 Democrats, it being understood that at least one Republican and one Democrat must be on the committee?

**15.** From a basket of 8 apples, in how many ways can a selection be made, it being understood that any or all of the apples can be taken?

**16.** How many triangles can be formed by joining the angular points of a decagon, that is, each triangle having its three vertices at vertices of the decagon?

**17.** There are 20 points in a plane, no three of which are in the same straight line with the exception of 5, which are all in one line. Find the number of straight lines that result from joining them.

**18.** Find the number of triangles which can be formed by joining the points mentioned in Ex. 17.

**19.** A boat's crew consists of 8 men, 3 of whom can row only on the port side and 2 of whom can row only on the starboard side. In how many ways may the crew be seated?

**20.** Out of 6 friends, in how many ways can you invite one or more of them to dinner?

**21.** From 3 peaches, 4 apples and 2 oranges, how many selections of fruit can be made, taking at least one of each kind?

# CHAPTER XIII

## PROBABILITY

**107. Introduction.** If a letter be chosen at random from the alphabet the chance, or probability, that it will be *a* is naturally regarded as 1/26 since, out of the 26 ways in which a letter may be drawn, only 1 gives *a*. Similarly, the probability, or chance, of drawing any single letter, as *m*, would be 1/26. However, if we ask the probability of drawing a *vowel*, the answer would be 5/26, since a vowel may be drawn in any one of 5 ways; namely, either *a*, *e*, *i*, *o* or *u*.

As another example, suppose that a bag contains 4 red balls and 5 white balls, and that a ball is drawn at random. The probability that it will be *red* is then 4/9, since out of the total of 9 ways of drawing a ball, 4 give red ones. Similarly, the probability of drawing a *white* ball is 5/9. These and other illustrations which may be readily supplied lead to the following definition.

DEFINITION. *The probability of an event is the ratio of the number of ways in which it can happen (all regarded as equally likely) to the total number of ways in which it can either happen or fail.*

Thus the probability of drawing an ace from a pack of cards is 4/52, or 1/13, since there are 4 ways in which the event can happen out of a total of 52 ways in which it can either happen or fail, the latter being the total number of cards in the pack.

This definition, when stated in algebraic language, is as follows. Let $a$ be the number of ways in which an event can happen, and let $b$ be the number of ways in which it can fail (all ways being regarded as equally likely). Then the probability, $p$, that the event will happen is

$$p=\frac{a}{a+b}. \tag{1}$$

COROLLARY 1. *If an event is certain to happen, its probability is* 1. For in (1) we then have $b=0$, giving $p=a/a=1$.

COROLLARY 2. *The probability that an event will happen and the probability that it will fail, when added together, give* 1. For, just as the fraction (1) is the probability that the event will *happen*, so the fraction

$$q=\frac{b}{a+b}. \tag{2}$$

is the probability that the event will *fail*, and it is evident that the sum of the expressions (1) and (2) is 1.

**108. Value of an Expectation.** If a person is to receive \$100 in case a certain event happens, and the probability of the event is 3/5, then the value of his expectation is naturally $3/5\times 100=\$60$. This amount, in other words, is what he should pay for the privilege of being the possible recipient of the \$100. In general, we thus adopt the following definition.

DEFINITION. *If a person is to receive the sum of money $M$ in case an event occurs whose probability is $p$, then the value of his expectation is $pM$.*

### EXERCISES

**1.** A bag contains 6 red balls, 4 white balls and 3 blue balls. If a ball be drawn at random, what is the probability that it will be (*a*) red, (*b*) white, (*c*) blue?

**2.** From a suit of 13 hearts, 3 cards are drawn. What is the chance that they will be the ace, king and queen?

[HINT. Three cards may be drawn in ${}_{13}C_3$ ways.]

**3.** The four capital letters $A$, $B$, $C$, $D$ and the four small letters $a$, $b$, $c$, $d$ are shaken together in a hat after which three letters are drawn out at random. What is the probability that they will all be capitals?

SOLUTION. Since there are 8 letters in all, the total number of ways of drawing 3 letters of *any* kind is

$${}_8C_3=\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}=56.$$

Similarly, the number of ways of drawing 3 *capital* letters is

$$ {}_4C_3 = \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4. $$

Hence the desired probability is $\frac{4}{56}$, or $\frac{1}{14}$. *Ans.*

**4.** Find the probability in Ex. 3 that the three letters drawn shall consist of two capitals and 1 small letter.

[HINT. The number of ways of drawing 2 capitals and 1 small letter is ${}_4C_2 \times {}_4C_1$. (§§ 104, 101).]

**5.** A portfolio contains 15 bills, 6 of which are \$5 bills, 4 are \$2 bills and 5 are \$1 bills. If 4 bills be taken at random find the chance that

(*a*) all are \$5 bills,
(*b*) 3 are \$2 bills and 1 is a \$5 bill,
(*c*) all are \$1 bills.

**6.** A history of Rome in four volumes is placed on a library shelf at random. What is the probability that the volumes will be in their correct order: I, II, III, IV?

**7.** If 4 cards be drawn from an ordinary pack, what is the probability that

(*a*) they will all be hearts?
(*b*) that there will be 1 card of each suit?

**8.** If two tickets be drawn from a package of 20 tickets marked 1, 2, 3, ⋯, 20, what is the probability that both will be marked with odd numbers?

**9.** If three coins be tossed, what is the probability that
(*a*) all will be heads?
(*b*) there will be exactly two heads?
(*c*) there will be at least two heads?

**10.** If three cards be drawn from a pack, what is the probability that they will be an ace, king and queen of different suits?

**11.** A person is to receive \$5 in case he tosses two coins and they both come up heads. What is the value of his expectation?

**12.** What can a person properly afford to pay for the privilege of receiving \$7.50 in case that he draws 2 tickets from a box containing tickets marked from 1 to 15 inclusive and finds that the one is odd and the other even?

**109. Definitions.** The preceding discussions and illustrations of the theory of probability are the immediate consequences of the definition of the term "probability," as given in § 107. If one is to proceed farther into the subject, it is desirable to make certain fundamental distinctions between the possible kinds of events, as indicated below.

Two or more events are called *dependent* or *independent* according as the happening of one of them *does* or *does not* affect the happening of the others.

Thus, if a drawing be made at random of one letter from a box containing the letters $a$, $b$, $c$, $d$, $e$ and this be followed by another similar drawing, the two events would be *independent* in case the letter first drawn was replaced in the box before the second drawing, while the events would be *dependent* in case this was not done.

**110. Theorem Concerning Independent Events.** In determining the probability that two or more independent events will all happen, one may employ the following theorem.

THEOREM. *The probability that two or more independent events will all happen is equal to the* **product** *of their respective probabilities.*

Thus, suppose that two coins are tossed. The probability that the one will come up heads is evidently $1/2$, and the probability that the other will come up heads is likewise $1/2$. Therefore, the probability that *both* will come up heads is, by the above Theorem, $1/2 \times 1/2 = 1/4$.

This result may be verified by noting that the total number of ways in which the two coins may fall is $2 \times 2 = 4$, and of these only 1 gives two heads. Hence, the answer, as before, is $1/4$.

Similarly, the probability that three coins will all come up heads is $1/2 \times 1/2 \times 1/2 = 1/8$.

PROOF OF THEOREM. Suppose that the probabilities of the separate events are respectively $p_1$, $p_2$, $p_3$, $\cdots$, $p_r$, and let $a_1$ be the number of ways in which the event corresponding to $p_1$ can happen and $b_1$ the number of ways in which this event can fail; similarly let $a_2$ be the number of ways in which the event corresponding to $p_2$ can happen, and $b_2$ the number of ways in which this event can fail, etc. Then, by the definition stated in § 107, we shall have

$$(1) \qquad p_1=\frac{a_1}{a_1+b_1},\quad p_2=\frac{a_2}{a_2+b_2},\quad \cdots,\quad p_r=\frac{a_r}{a_r+b_r}.$$

Moreover, by the principle of § 101, all the separate events can happen together in $a_1 \cdot a_2 \cdot a_3 \cdots a_r$ ways out of

$$(a_1+b_1)(a_2+b_2)\cdots(a_r+b_r)$$

possible ways of either happening or failing. Hence, if $P$ be the probability that all the events will happen, we have by the definition in § 107,

$$(2) \qquad P=\frac{a_1a_2\cdots a_r}{(a_1+b_1)(a_2+b_2)\cdots(a_r+b_r)}.$$

But, upon using (1), the expression (2) may be written in the form

$$P=p_1p_2\cdots p_r,$$

which was to be proved.

**111. Dependent Events.** Although the theorem of § 110 pertains only to independent events, it may frequently be applied to determine probability in the case of dependent events, since the latter may usually be separated so as to be regarded as independent.

Example. One letter is drawn from a box containing the letters $a$, $b$, $c$, $d$, $e$ and a second drawing is then made (the first letter obtained *not* being replaced before the second drawing). What is the probability that the letters thus drawn are first $a$ and second $b$?

Solution. The probability of obtaining $a$ on the first drawing is evidently 1/5 and, $a$ having been drawn, the probability of obtaining $b$ on the second drawing is 1/4, since but 4 letters remain after the first drawing. Therefore, by the theorem of § 110, the desired probability is $1/5\times1/4=1/20$. *Ans.*

This result may be verified as follows. The total number of ways of drawing 2 letters is $5\times4=20$ and of these there is but one that gives first $a$ and then $b$. Hence, the probability is 1/20, which agrees with the former result.

**112. Theorem Concerning Events That Can Happen in Several Ways.** In determining the probability that an event will happen in case it can happen in any one of two or more different ways which are mutually exclusive, one may employ the following theorem.

THEOREM. *If an event can happen in any one of two or more different ways which are mutually exclusive, the probability that it will happen is the* **sum** *of the probabilities of its happening in these different ways.*

Thus, if it be asked what is the probability of getting either two heads or two tails when two coins are tossed, we may reason as follows. The probability of getting two heads, as shown in § 110, is 1/4, and similarly the probability of getting two tails is 1/4. Therefore, by the theorem above, the probability of getting *either* two heads or two tails is $1/4+1/4=1/2$.

This result may be verified by noting that the total number of ways in which the two coins may fall is $2\times 2=4$, and of these 1 gives both heads and 1 gives both tails. Hence, the probability of getting either both heads or both tails is $\frac{1+1}{4}=2/4=1/2$, thus agreeing with the former result.

PROOF OF THEOREM. Suppose that the event can happen in *two* mutually exclusive ways, and let $p_1=a_1/b_1$ and $p_2=a_2/b_2$ be their respective probabilities. Then, out of a total of $b_1\cdot b_2$ possible cases leading to success or failure in either of the two ways, there are $a_1 b_2$ in which the event can happen in the first way and $a_2b_1$ in which it can happen in the second way. Hence, out of the $b_1b_2$ cases there are $a_1b_2+a_2b_1$ cases in which the event can happen in the one or the other of the two ways, the probability of which is therefore

$$\frac{a_1b_2+a_2b_1}{b_1b_2}=\frac{a_1}{b_1}+\frac{a_2}{b_2}=p_1+p_2.$$

The theorem thus becomes proved in case there are but *two* ways in which the event can happen. Similar reasoning leads directly to the more general case.

**113. Theorem Concerning Repeated Trials.** If the probability of the happening of an event in a single trial is known, the probability that it will happen exactly $r$ times in $n$ trials may be determined by use of the following theorem.

THEOREM. *If $p$ is the probability that an event will happen in any single trial, then the probability that it will happen exactly $r$ times in $n$ trials is ${}_nC_rp^rq^{n-r}$, where $q$ is the probability that the event will fail in any one trial.*

Thus, if it be asked what is the probability of throwing exactly 3 aces in 5 throws with a single die, the answer is

$${}_5C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2 = 10 \cdot \frac{1}{216} \cdot \frac{25}{36} = \frac{125}{3888}.$$

PROOF OF THEOREM. The probability that the event will happen in $r$ specified trials and fail in the remaining $(n-r)$ trials is, by § 110, $p^rq^{n-r}$. But the $r$ trials can be selected out of the $n$ trials in ${}_nC_r$ ways. Hence, applying the theorem of § 112, it follows that the probability in question is the result of adding $p^rq^{n-r}$ to itself ${}_nC_r$ times; that is, it is equal to

$${}_nC_r\, p^rq^{n-r}.$$

It is to be observed that if we expand $(p+q)^n$ by the Binomial Theorem (§ 104, Note), we obtain

$$p^n + {}_nC_1p^{n-1}q + {}_nC_2p^{n-2}q^2 + \cdots + {}_nC_{n-r}p^rq^{n-r} + \cdots + q^n.$$

Thus, the terms of this expansion represent respectively the probabilities of the happening of the event exactly $n$ times, $(n-1)$ times, $(n-2)$ times, $\cdots$ in $n$ trials.

Moreover, by combining this result with that of § 112, we obtain the following corollary.

COROLLARY *The probability that an event will happen* ***at least*** *$r$ times in $n$ trials is*

$$p^n + {}_nC_1p^{n-1}q + {}_nC_2p^{n-2}q^2 + \cdots + {}_nC_{n-r}p^rq^{n-r},$$

*where $p$ and $q$ have the meanings indicated above.* In fact, by § 112, this expression comes to represent the probability that the event will happen either exactly $n$ times, or exactly $(n-1)$ times, or exactly $(n-2)$ times, $\cdots$ or exactly $r$ times; that is, that it will happen *at least r times.*

Thus the probability of obtaining *at least* 3 aces in 5 throws with a single die is

$$\left(\frac{1}{6}\right)^5+{}_5C_1\cdot\left(\frac{1}{6}\right)^4\cdot\left(\frac{5}{6}\right)+{}_5C_2\cdot\left(\frac{1}{6}\right)^3\cdot\left(\frac{5}{6}\right)^2=\frac{1+5\cdot 5+10\cdot 25}{6^5}=\frac{276}{6^5}=\frac{23}{648}.$$

## EXERCISES

**1.** Find the probability of throwing an ace in the first only of two successive throws with a die.

**2.** If three cards be drawn from a pack, find the probability that they will be an ace, a king and a queen in the order named.

**3.** Work Ex. 2 in case no regard is had for the *order* in which the three desired cards are obtained.

[HINT. Consider each possible order and apply the theorem of § 112 to the separate results.]

**4.** Find, by use of the theorem of § 112, the probability of throwing doublets in a single throw with a pair of dice.

**5.** In three throws with a pair of dice, find the probability of throwing doublets at least once.

**6.** A bag contains 5 white and 3 black balls, and 4 are successively drawn out and not replaced. What is the probability that they are alternately of different colors?

**7.** A, B, C in the order named each draw a card from an ordinary pack, replacing the drawing each time. If the first one to obtain a spade is to win a prize, show that their expectations are in the ratio 16:12:9.

[HINT. First find the probability that A obtains a spade; second, the probability that A fails to obtain a spade, but B obtains one; etc. It is understood that a total of only three drawings can be made.]

**8.** A and B throw with one die for a stake which is to be won by the player who first throws an ace. A has the first throw and the throwing is to continue alternately until either the one or the other wins. Show that their respective probabilities of winning are

$$\frac{1}{6}\left\{1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\cdots\right\}\text{ and }\frac{5}{6}\cdot\frac{1}{6}\left\{1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\cdots\right\}$$

and hence that their respective expectations are in the ratio of 6:5.

**114. Probability of Human Life. Mortality Table.** If a person 19 years of age asks what the probability is that he will live to the age of 75, the question may be answered with good accuracy by consulting a so-called *Experience Table of Mortality*. Such a table is shown on the opposite page and is readily understood upon examination. It shows in particular that out of 93,362 persons living at the age of 19 it may be expected that at the age of 75 there will remain 26,237. Hence, the answer to the preceding question is 26,237/93,362, or about 0.28 Otherwise stated, the chances that a person of 19 will live to be 75 are about 28 out of 100.

The table on page 203 was compiled from the averaged observations of thirty American insurance companies to the end of the year 1874. Such a table is evidently of vital importance in answering the questions which ordinarily come before a life insurance company, or any person entrusted to work out a proper pension system for a group of employees, or the judge who wishes to determine what is a proper life interest of a client in an estate. Such questions depend upon the probable extent of life of an individual at a given age.

### EXERCISES

**1.** What is the probability that the average American boy of 10 years will live to vote at a public election? What is the probability that he will live to the age of 80?

**2.** A man is 47 and his son is 15. Show that the probability that both will live 10 years is about 0.79.

[HINT. Apply the theorem of § 110.]

**3.** A bridegroom of 24 marries a bride of 21. Show that the probability that they will live to celebrate their golden wedding is about 0.12

**4.** A and B are twins just 18 years old. Show that the probability that both will attain the age of 50 is about 0.55; also, that the probability that one, but not both, will die before the age of 50 is about 0.38.

[HINT. Employ the theorem of § 112.]

**5.** Draw on coördinate paper the graph of the curve showing the probability of dying for each year from the ages of 10 to 90.

## AMERICAN EXPERIENCE TABLE OF MORTALITY

| AGE | NUMBER LIVING | NUMBER DYING | AGE | NUMBER LIVING | NUMBER DYING | AGE | NUMBER LIVING | NUMBER DYING |
|---|---|---|---|---|---|---|---|---|
| $x$ | $l_x$ | $d_x$ | $x$ | $l_x$ | $d_x$ | $x$ | $l_x$ | $d_x$ |
| **10** | 100 000 | 749 | **40** | 78 106 | 765 | **70** | 38 569 | 2391 |
| **11** | 99 251 | 746 | **41** | 77 341 | 774 | **71** | 36 178 | 2448 |
| **12** | 98 505 | 743 | **42** | 76 567 | 785 | **72** | 33 730 | 2487 |
| **13** | 97 762 | 740 | **43** | 75 782 | 797 | **73** | 31 243 | 2505 |
| **14** | 97 022 | 737 | **44** | 74 985 | 812 | **74** | 28 738 | 2501 |
| **15** | 96 285 | 735 | **45** | 74 173 | 828 | **75** | 26 237 | 2476 |
| **16** | 95 550 | 732 | **46** | 73 345 | 848 | **76** | 23 761 | 2431 |
| **17** | 94 818 | 729 | **47** | 72 497 | 870 | **77** | 21 330 | 2369 |
| **18** | 94 089 | 727 | **48** | 71 627 | 896 | **78** | 18 961 | 2291 |
| **19** | 93 362 | 725 | **49** | 70 731 | 927 | **79** | 16 670 | 2196 |
| **20** | 92 637 | 723 | **50** | 69 804 | 962 | **80** | 14 474 | 2091 |
| **21** | 91 914 | 722 | **51** | 68 842 | 1001 | **81** | 12 383 | 1964 |
| **22** | 91 192 | 721 | **52** | 67 841 | 1044 | **82** | 10 419 | 1816 |
| **23** | 90 471 | 720 | **53** | 66 797 | 1091 | **83** | 8 603 | 1648 |
| **24** | 89 751 | 719 | **54** | 65 706 | 1143 | **84** | 6 955 | 1470 |
| **25** | 89 032 | 718 | **55** | 64 563 | 1199 | **85** | 5 485 | 1292 |
| **26** | 88 314 | 718 | **56** | 63 364 | 1260 | **86** | 4 193 | 1114 |
| **27** | 87 596 | 718 | **57** | 62 104 | 1325 | **87** | 3 079 | 933 |
| **28** | 86 878 | 718 | **58** | 60 779 | 1394 | **88** | 2 146 | 744 |
| **29** | 86 160 | 719 | **59** | 59 385 | 1468 | **89** | 1 402 | 555 |
| **30** | 85 441 | 720 | **60** | 57 917 | 1546 | **90** | 847 | 385 |
| **31** | 84 721 | 721 | **61** | 56 371 | 1628 | **91** | 462 | 246 |
| **32** | 84 000 | 723 | **62** | 54 743 | 1713 | **92** | 216 | 137 |
| **33** | 83 277 | 726 | **63** | 53 030 | 1800 | **93** | 79 | 58 |
| **34** | 82 551 | 729 | **64** | 51 230 | 1889 | **94** | 21 | 18 |
| **35** | 81 822 | 732 | **65** | 49 341 | 1980 | **95** | 3 | 3 |
| **36** | 81 090 | 737 | **66** | 47 361 | 2070 | | | |
| **37** | 80 353 | 742 | **67** | 45 291 | 2158 | | | |
| **38** | 79 611 | 749 | **68** | 43 133 | 2243 | | | |
| **39** | 78 862 | 756 | **69** | 40 890 | 2321 | | | |

# CHAPTER XIV

## DETERMINANTS

**115. Definitions.** The symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a ***determinant of the second order*** and is defined as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Thus

$$\begin{vmatrix} 8 & 3 \\ 2 & 4 \end{vmatrix} = 8 \cdot 4 - 2 \cdot 3 = 32 - 6 = 26,$$

$$\begin{vmatrix} 7 & 3 \\ -2 & 4 \end{vmatrix} = 7 \cdot 4 - (-2) \cdot 3 = 28 + 6 = 34,$$

The numbers $a$, $b$, $c$, and $d$ are called the ***elements*** of the determinant.

The elements $a$ and $d$ (which lie along the diagonal through the upper left-hand corner of the determinant) form the ***principal diagonal.*** The letters $b$ and $c$ (which lie along the diagonal through the upper right-hand corner) form the ***minor diagonal.***

From these definitions, we have the following rule.

*To evaluate any determinant of the second order, subtract the product of the elements in the minor diagonal from the product of the elements in the principal diagonal.*

### EXERCISES

Evaluate each of the following determinants.

**1.** $\begin{vmatrix} 8 & 2 \\ 3 & 1 \end{vmatrix}$.

**2.** $\begin{vmatrix} 5 & -1 \\ 7 & 3 \end{vmatrix}$.

**3.** $3\begin{vmatrix} -1 & -4 \\ 3 & -5 \end{vmatrix}$.

**4.** $\begin{vmatrix} 2a & 3b \\ 4a & 5b \end{vmatrix}$.

**5.** $\frac{2}{3}\begin{vmatrix} 3a & 0 \\ 6b & 1 \end{vmatrix}$.

**6.** $\frac{3}{4}\begin{vmatrix} a^2+b^2 & 4 \\ a^2-b^2 & 4 \end{vmatrix}$.

**116. Solution of Two Linear Equations.** Let us consider a system of two linear equations between two unknown letters, $x$ and $y$. Any such system is of the form

$$a_1x+b_1y=c_1, \tag{1}$$
$$a_2x+b_2y=c_2, \tag{2}$$

where $a_1$, $b_1$, $c_1$, etc., represent known numbers (coefficients).

This system may be solved for $x$ and $y$ by elimination, as in § 5. Thus, multiplying (1) by $b_2$ and (2) by $b_1$, subtracting the resulting equations from each other, and solving for $x$, we find

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}. \tag{3}$$

Likewise, we may eliminate $x$ by multiplying (1) by $a_2$ and (2) by $a_1$. Subtracting the resulting equations from each other and solving for $y$, we find

$$y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}. \tag{4}$$

It is now clear, by § 115, that the numerators and denominators in (3) and (4) are all determinants of the second order; and by the definition of § 115, (3) and (4) may be written respectively in the forms

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{5}$$

These forms are easily remembered. Observe that:

1. The determinant for the denominator is the same for both $x$ and $y$.

2. The determinant for the numerator of the $x$-value is the same as that for the denominator except that the numbers $c_1$ and $c_2$ replace the $a_1$ and $a_2$ which occur in the *first* column of the denominator determinant.

3. The determinant for the numerator of the $y$-value is the same as that for the denominator except that the numbers $c_1$ and $c_2$ replace the $b_1$ and $b_2$ which occur in the *second* column of the denominator determinant.

The usefulness of the forms (5) lies in the fact that they express the solution of a system of two linear equations in condensed form, enabling us to write down the desired values of $x$ and $y$ *immediately*, without the usual process of elimination. This will now be illustrated.

Example. Solve by determinants the system

$$2x+3y=18, \tag{6}$$
$$x-7y=-8. \tag{7}$$

Solution. Using the forms (5), we have at once

$$x=\frac{\begin{vmatrix}18 & 3\\ -8 & -7\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 1 & -7\end{vmatrix}}=\frac{18\cdot(-7)-(-8)\cdot 3}{2(-7)-1\cdot 3}=\frac{-126+24}{-14-3}=\frac{-102}{-17}=6,$$

$$y=\frac{\begin{vmatrix}2 & 18\\ 1 & -8\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 1 & -7\end{vmatrix}}=\frac{2\cdot(-8)-1\cdot 18}{2(-7)-1\cdot 3}=\frac{-16-18}{-14-3}=\frac{-34}{-17}=2.$$

The solution desired is therefore $(x=6,\ y=2)$. *Ans.*

Check. Substituting 6 for $x$ and 2 for $y$ in (6) and (7) gives $12+6=18$ and $6-14=-8$, which are true results.

## EXERCISES

Solve each of the following pairs of equations by determinants, checking your answers for each of the first three.

**1.** $\begin{cases}2x-3y=10,\\ 5x+2y=6.\end{cases}$

**2.** $\begin{cases}5x+y=22,\\ x+5y=14.\end{cases}$

**3.** $\begin{cases}3x+8y=0,\\ 2x-9y=-11.\end{cases}$

**4.** $\begin{cases}x+\frac{1}{3}y=11,\\ \frac{1}{3}x+3y=21.\end{cases}$

**5.** $\begin{cases}1-x=3y,\\ 3-3x=40-y.\end{cases}$

**6.** $\begin{cases}ax+by=m,\\ bx-ay=c.\end{cases}$

**7.** $\begin{cases}x-ay=n,\\ bx+y=p.\end{cases}$

**8.** $\begin{cases}x+y=b-a,\\ bx-ay=-2ab.\end{cases}$

**9.** $\begin{cases}3ax+2by=ab,\\ ax-by=ab.\end{cases}$

**117. Determinants of the Third Order.** The symbol

$$(1) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a ***determinant of the third order.***

Its value is defined as follows:

$$(2) \qquad a_1b_2c_3+b_1c_2a_3+c_1a_2b_3-a_3b_2c_1-b_3c_2a_1-c_3a_2b_1.$$

This expression, as we shall see presently, is important in the study of equations.

The expression (2) is called the *expanded form* of the determinant (1). It is important to observe that this expanded form may be written down at once as follows.

Write the determinant with the first two columns repeated at the right and first note the three diagonals which then run *down* from left to right (marked +). The product of the elements in the first of these diagonals is $a_1b_2c_3$, and this is seen to be the first term of the expanded form (2). Similarly, the product of the elements in the second of these diagonals is $b_1c_2a_3$, which forms the second term of (2); and likewise the third diagonal furnishes at once the third term of (2).

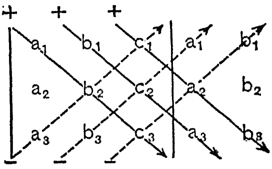

Fig. 57

Next consider the three diagonals which run *up* from left to right (marked with dotted lines). The product of the elements in the first of these is $a_3b_2c_1$, and this is the fourth term of (2), provided it be taken negatively, that is, preceded by the sign −. Similarly, the other two dotted diagonals of (3) furnish the last two terms of (2), provided they be taken negatively.

**Note.** Every determinant of the third order when expanded contains a total of *six* terms.

Example. Expand and find the value of the determinant

$$\begin{vmatrix} 3 & 7 & 9 \\ 2 & 1 & 4 \\ 6 & 3 & 2 \end{vmatrix}$$

Solution. Repeating the first and second columns at the right, we have

$$\left|\begin{array}{ccc|cc} 3 & 7 & 9 & 3 & 7 \\ 2 & 1 & 4 & 2 & 1 \\ 6 & 3 & 2 & 6 & 3 \end{array}\right|.$$

The diagonals running down from left to right give the three products

$$3 \cdot 1 \cdot 2, \quad 7 \cdot 4 \cdot 6, \quad 9 \cdot 2 \cdot 3,$$

which form the first three terms of the expansion.

The diagonals running up from left to right give the products

$$6 \cdot 1 \cdot 9, \quad 3 \cdot 4 \cdot 3, \quad 2 \cdot 2 \cdot 7,$$

which, when taken negatively, form the three remaining terms of the determinant.

The complete expanded form of (3) is, therefore,

$$3 \cdot 1 \cdot 2 + 7 \cdot 4 \cdot 6 + 9 \cdot 2 \cdot 3 - 6 \cdot 1 \cdot 9 - 3 \cdot 4 \cdot 3 - 2 \cdot 2 \cdot 7,$$

which reduces to

$$6 + 168 + 54 - 54 - 36 - 28 = 110. \quad \textit{Ans.}$$

## EXERCISES

Expand and find the value of each of the following determinants.

**1.** $\begin{vmatrix} 1 & 2 & 7 \\ 2 & 2 & 6 \\ 3 & 2 & -4 \end{vmatrix}.$

**2.** $\begin{vmatrix} -7 & 4 & 2 \\ 3 & 2 & 6 \\ 8 & -8 & -3 \end{vmatrix}.$

**3.** $\begin{vmatrix} 8 & 2 & 3 \\ 3 & 0 & -2 \\ 3 & 0 & 7 \end{vmatrix}.$

**4.** $\begin{vmatrix} 6 & 9 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix}.$

**5.** $\begin{vmatrix} x & 7 & 8 \\ 2 & 3 & -1 \\ 4 & 2 & 3 \end{vmatrix}.$

**6.** $\begin{vmatrix} a & b & 2 \\ -2 & 6 & 3 \\ 2 & 1 & 0 \end{vmatrix}.$

**7.** $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix}.$

**8.** $\begin{vmatrix} 1 & 0 & 0 \\ 0 & x-y & 0 \\ 0 & 0 & x+y \end{vmatrix}.$

**118. Solution of Three Linear Equations.** Let us consider a system of three linear equations between three unknown letters, such as $x$, $y$, and $z$. Any such system is of the form

$$(1) \qquad \begin{cases} a_1x+b_1y+c_1z=d_1, \\ a_2x+b_2y+c_2z=d_2, \\ a_3x+b_3y+c_3z=d_3, \end{cases}$$

where $a_1$, $b_1$, $c_1$, $d_1$, $a_2$, $b_2$, etc., represent known numbers (coefficients).

This system may be solved for $x$, $y$, and $z$ by elimination, as in § 5, but the process is long. We shall here state merely the results, which are as follows (compare with (3) and (4) of § 116):

$$(2) \qquad \begin{cases} x=\dfrac{d_1b_2c_3+d_2b_3c_1+d_3b_1c_2-d_3b_2c_1-d_1b_3c_2-d_2b_1c_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3}, \\ y=\dfrac{a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_1d_3c_2-a_2d_1c_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3}, \\ z=\dfrac{a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_1b_3d_2-a_2b_1d_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3}. \end{cases}$$

It is clear by § 117 that in these values for $x$, $y$, and $z$, each numerator and denominator is the expanded form of a determinant of the third order. In fact, it appears from the definition in § 117, that we may now express these values of $x$, $y$, and $z$ in the following condensed (determinant) forms:

$$(3) \quad x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

NOTE. The importance of these expressions for $x$, $y$, and $z$ lies in the fact that they give at once the solution of any system such as (1) in very compact and easily remembered forms. Here we note that:

1. The denominator determinant is the same in all three cases. (Compare statement 1 of § 116.)

2. The determinant for the numerator of the $x$-value is the same as that for the denominator determinant except that the numbers $d_1$, $d_2$, $d_3$ replace the $a_1$. $a_2$, $a_3$ which occur in the *first* column of the denominator determinant.

3. Similarly, the numerator of the $y$-value is formed from that of the denominator determinant by replacing the *second* column by the elements $d_1$, $d_2$, $d_3$; while the numerator of the $z$-value is formed from that of the denominator determinant by replacing the *third* column by the elements $d_1$, $d_2$, $d_3$. (Compare statements 2 and 3 of § 116.)

The readiness with which (3) may be used in practice to solve a system of three linear equations is illustrated below.

Example. Solve the system

$$\begin{cases} 2x-y+3z=35, \\ x+3y-15=-2z, \\ \quad 3x+4y=1. \end{cases}$$

Solution. Arranging the equations as in (1) of § 118, the given system is

$$\begin{aligned} 2x-y+3z&=35, \\ x+3y+2z&=15, \\ 3x+4y+0z&=1. \end{aligned}$$

Therefore, using (3) of § 118, we have at once

$$x=\frac{\begin{vmatrix} 35 & -1 & 3 \\ 15 & 3 & 2 \\ 1 & 4 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 3 & 4 & 0 \end{vmatrix}}=\frac{0+180-2-9-280-0}{0+12-6-27-16-0}=\frac{-111}{-37}=3, \qquad (\S\ 117)$$

$$y=\frac{\begin{vmatrix} 2 & 35 & 3 \\ 1 & 15 & 2 \\ 3 & 1 & 0 \end{vmatrix}}{-37}=\frac{0+3+210-135-4-0}{-37}=\frac{74}{-37}=-2,$$

$$z=\frac{\begin{vmatrix} 2 & -1 & 35 \\ 1 & 3 & 15 \\ 3 & 4 & 1 \end{vmatrix}}{-37}=\frac{6+140-45-315-120+1}{-37}=\frac{-333}{-37}=9.$$

The desired solution is, therefore, $(x=3,\ y=-2,\ z=9)$. *Ans.*

CHECK. With $x=3$, $y=-2$, $z=9$, it is readily seen that the three given equations are satisfied.

### EXERCISES

Solve each of the following systems by determinants.

**1.** $\begin{cases} x+2y+3z=14, \\ 2x+y+2z=10, \\ 3x+4y-3z=2. \end{cases}$

**2.** $\begin{cases} 2x-y+2z=12, \\ x+3y+z=41, \\ 2x+y+4z=22. \end{cases}$

**3.** $\begin{cases} x-y+z=30, \\ 3y-x-z=12, \\ 7z-y+2x=141. \end{cases}$

**4.** $\begin{cases} x+3y+4z=83, \\ x+y+z=29, \\ 6x+8y+3z=156. \end{cases}$

**5.** $\begin{cases} 3x-2y+z=2, \\ 2x+5y+2z=27, \\ x+3y+3z=25. \end{cases}$

**6.** $\begin{cases} x+y=9, \\ y+z=7, \\ z+x=5. \end{cases}$

**7.** $\begin{cases} x+y-z=0, \\ x-y=2b, \\ x+z=3a+b. \end{cases}$

**8.** $\begin{cases} ax+by+cz=3, \\ abx+aby=a+b, \\ bcy+bcz=b+c. \end{cases}$

**119. Determinants of Higher Order.** The determinants thus far studied have been of either the second or third orders, the former containing $2^2$, or 4 elements, and the latter $3^2$, or 9 elements. In general, a determinant of the $n$th order is a square array of $n^2$ elements such as is typified by the expression

$$D=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \cdots l_1 \\ a_2 & b_2 & c_2 & d_2 \cdots l_2 \\ a_3 & b_3 & c_3 & d_3 \cdots l_3 \\ \cdots & \cdots & \cdots & \cdots\cdots\cdots \\ a_n & b_n & c_n & d_n \cdots l_n \end{vmatrix}.$$

The method for obtaining the expanded form of any such determinant (compare (2), § 117) will be explained in detail in § 121.

**120. Inversions of Order** Consider the positive integers 1, 2, 3, 4. As here appearing, these are in their *natural* order, each number being less than all those which follow it. If the same numbers be arranged as follows: 4, 2, 3, 1, there are five departures from the natural order; namely, 4 before 2, 4 before 3, 4 before 1, 2 before 1 and 3 before 1. Each of these is called an ***inversion of order,*** Briefly stated, we say that 4, 2, 3, 1 contains five inversions.

Similarly, any given arrangement of two or more positive integers contains a certain number of inversions, this number being 0 only in case the numbers occur in their natural order.

Thus, in 3, 4, 1, 2, there are 4 inversions; namely, 3 before 1, 3 before 2, 4 before 1 and 4 before 2. Similarly, in 1, 3, 4, 5, 2 there are 3 inversions; in 1, 3, 2 there is 1 inversion, etc.

**121. The Expanded Form of Any Determinant.** An examination of the expanded form of the typical determinant of the *third* order, as given in (2) of § 117, shows that it may be written in the form

$$a_1b_2c_3+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3.$$

It is now to be observed that each of the six terms here appearing contains three factors, of which the first is an $a$, the second a $b$ and the third a $c$, and the subscripts of these letters in any one term are all different, as for example in the third term $a_2b_3c_1$. Moreover, in the case of the three terms which are preceded by the sign $+$, the number of inversions in the subscripts is *even,* while in the case of the three terms preceded by the sign $-$, the number of inversions in the subscripts is *odd.*

Thus, in the term $+a_3b_1c_2$, there are *two* inversions in the subscripts, this number being *even,* while in the term $-a_3b_2c_1$ there are *three* such inversions, this being *odd.* Similarly, the term $+a_2b_3c_1$, is seen to be accompanied with an even number of inversions, while $-a_2b_1c_3$ has an odd number of them.

Taking now the type determinant of the *fourth* order, namely

$$(1)\qquad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

the observations made above suggest that its expanded form consists of all the terms that can be made, each consisting of *four* factors of which the first is an $a$, the second a $b$, the third a $c$ and the fourth a $d$ and in which no two subscripts are alike, and with the further understanding that the sign to be prefixed to any one term as thus formed is to be $+$ or $-$ according as the number of inversions in its subscripts is even or odd. This, in fact, is what the meaning of (1) is taken to be, and we shall so understand hereafter.

For example, $+a_1b_2c_3d_4$, $-a_2b_3c_4d_1$, $+a_2b_3c_1d_4$ are three particular terms in the expansion of (1).

It may be observed that the total number of terms as thus described belonging to the expanded form of (1) is 24, or 4!, since the $a$ to be used in forming a term may first be chosen in any one of 4 ways, then the $b$ may be chosen in any one of 3 ways (its subscript being necessarily different from that of the $a$ chosen), then the $c$ may be chosen in any one of 2 ways (its subscript being neither of those already used), and finally the $d$ may be chosen in but 1 way and therefore, by § 101, the four elements for any one term may be selected in $4 \cdot 3 \cdot 2 \cdot 1 = 24 = 4!$ ways. The student is advised to write out all of the 24 terms, prefixing the proper sign to each.

Similarly, the expanded form for the typical *fifth* order determinant may now be supplied. In this case there are $5! = 120$ terms each of the form $a_r\, b_s\, c_t\, d_u\, e_v$, where no two of the subscripts $r$, $s$, $t$, $u$, $v$ are alike and where the sign of any one term is taken $+$ or $-$ according as the number of inversions among these subscripts is even or odd.

Likewise, for any given value of $n$, the determinant $D$ of § 119 may be expanded, this expansion containing in all $n!$ terms, each the product of $n$ elements properly chosen.

**NOTE.** For convenience, the typical determinant of the third order, namely (1) of § 117, is frequently written in the condensed form $|a_1b_2c_3|$. Likewise, the typical fourth order determinant may be represented by $|a_1b_2c_3d_4|$, and similarly for determinants of higher orders.

### EXERCISES

**1.** Write, with their proper signs, all the terms of the determinant $|a_1b_2c_3d_4e_5|$ that contain both $a_1$ and $b_4$; also all the terms that contain both $b_3$ and $e_5$.

**2.** By expanding the following determinant, show that its value is 19.

$$\begin{vmatrix} 3 & 2 & 2 & 0 \\ 5 & 3 & 1 & 0 \\ 6 & 6 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{vmatrix}.$$

[HINT. Note that in the notation of § 119, the first column contains the $a$'s, the second column the $b$'s etc., so that we here have $a_1=3$, $a_2=5$, $a_3=6$, $a_4=0$, $b_1=2$, $b_2=3$, etc.]

**3.** Find, by expanding, the value of the determinant

$$\begin{vmatrix} 0 & 8 & 2 & 2 \\ 1 & 3 & 2 & 1 \\ 0 & -5 & -1 & 1 \\ 0 & 9 & 6 & 1 \end{vmatrix}.$$

**122. Useful Properties of Determinants.** The following theorems are useful in the study of determinants.

THEOREM I. *Two determinants are equal in case the elements of the first column of the one are equal respectively to the elements of the first row of the other, the elements of the second column of the one are equal respectively to the elements of the second row of the other, and so on.*

Thus
$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 2 & 0 & 3 \end{vmatrix}.$$

PROOF. Let us consider the theorem first for determinants of the third order. What we are then to prove is that

$$(1)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The determinant on the left side of (1) when expanded by the method of § 121, is equal to

$$(2)\qquad a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1.$$

As to the determinant on the right side of (1), if we place $A_1=a_1$, $A_2=b_1$, $A_3=c_1$, $B_1=a_2$, $B_2=b_2$, $B_3=c_2$, $C_1=a_3$, $C_2=b_3$, $C_3=c_3$, it becomes

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

and this, when expanded by the method of § 121, becomes

$$(3)\qquad A_1B_2C_3+A_2B_3C_1+A_3B_1C_2-A_1B_3C_2-A_2B_1C_3-A_3B_2C_1$$

If we replace $A_1$, $A_2$, $A_3$, $B_1$ ⋯ by their values as defined above, (3) becomes

$$(4)\qquad a_1b_2c_3+b_1c_2a_3+c_1a_2b_3-a_1c_2b_3-b_1a_2c_3-c_1b_2a_3$$

which is seen to be the same in value as (1), thus proving the theorem. Similarly, the proof may be given for determinants of any order.

THEOREM II. *If two rows (or columns) of a determinant are interchanged, the value of the new determinant thus obtained is the same as the original except that its sign is changed.*

Thus
$$\begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 3 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}.$$

Here the first and last rows of the original determinant have been interchanged in obtaining the new determinant.

PROOF. Consider first that we merely interchange two *adjacent* rows of any determinant. This will merely interchange two adjacent subscripts in each term of its expansion. This will change the sign of every term in the expansion, by § 121, and hence will change the sign of the whole determinant.

If, more generally, the two rows to be interchanged are separated by $m$ intermediate rows, we first note that the lower row may be brought just below the upper one by $m$ successive interchanges of adjacent rows. To bring the upper row into the original position of the lower one then requires $m+1$ further successive interchanges. It follows that interchanging the two rows in question is equivalent to introducing

$$m+(m+1)=2m+1$$

interchanges of adjacent rows and therefore, from what is said above, is equivalent to multiplying the original determinant $2m+1$ times by $-1$; that is, by $(-1)^{2m+1}$. But $2m+1$ is necessarily an odd number whatever the (positive, integral) value of $m$. Hence $(-1)^{2m+1}$ is equal in all cases to $-1$, so that the theorem becomes proved for the case of the interchange of any two rows. To prove it also for the case of the interchange of any two columns, it suffices to write the original determinant, as we may do by Theorem I, in a form where its successive rows and columns become interchanged and then apply to the result the reasoning already given concerning the interchange of two rows.

THEOREM III. *If a determinant D has two of its rows (or columns) identical, its value is zero.*

For example, without expanding the determinant, we may write at once

$$\begin{vmatrix} 1 & -1 & 3 & 4 \\ 2 & 5 & 3 & 1 \\ 1 & -1 & 3 & 4 \\ 5 & 6 & 8 & 7 \end{vmatrix} = 0,$$

the first and third rows being here identical.

Proof. By interchanging the two identical rows we obtain, by Theorem II, the value $-D$. But, interchanging two *identical* rows does not alter the form of the original determinant. Hence, we have $D=-D$, or $2D=0$, or $D=0$. Similarly, the proof for the case of the interchange of two identical columns follows directly from Theorem II.

Theorem IV. *If every element of a row (or column) of a determinant is multiplied by any given number m, the determinant is multiplied by m.*

Thus
$$\begin{vmatrix} 2 & 3 & 4 \\ -1 & 1 & 2 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 4 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 & 4 \\ -1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}.$$

Here the elements of the last row, regarded as 3, 2, 4, are each multiplied by 2.

Proof. The theorem is an immediate consequence of the fact that one and only one of the elements that have been multiplied by $m$ enters into each term of the expansion, thus multiplying the whole expansion by $m$.

Theorem V. *If each of the elements in a row (or column) is expressed as the sum of two numbers, the determinant may be expressed as the sum of two determinants.* That is, (in the case of the third order determinant)

$$\begin{vmatrix} a_1+a_1' & b_1 & c_1 \\ a_2+a_2' & b_2 & c_2 \\ a_3+a_3' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix}.$$

Proof. Consider any term of the expansion of the given determinant, as $(a_1+a_1')b_2c_3$. This may be written $a_1b_2c_3+a_1'b_2c_3$. Likewise, every term in the expanded form of the first determinant consists of the sum of a term of the second determinant and a term of the third determinant. Hence, the first determinant is the sum of the other two determinants.

Similarly, the proof can be supplied whatever the order of the given determinant.

THEOREM VI. *The value of a determinant is not changed if the elements in any row (or column) are multiplied by any number m, and added to, or subtracted from, the corresponding elements in any other row (or column).* Thus, for example,

$$\begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

PROOF. By Theorem V, the first determinant here appearing may be expressed as follows:

$$(1) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}.$$

But, the last determinant, by Theorem IV, may be written as

$$(2) \qquad m\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}.$$

and, applying Theorem III, this has the value $m \cdot 0 = 0$, with which the proof is complete for the third order determinant above considered.

Similarly, the proof may be supplied in all other cases.

**123. The Simplification of Determinants.** The theorems of § 122, especially Theorem VI, are of great value in reducing given determinants to simpler forms. The manner in which this is done will be clear from an examination of the following examples.

EXAMPLE 1.

$$\begin{vmatrix} 17 & 19 & 23 \\ 13 & 15 & 16 \\ 11 & 13 & 17 \end{vmatrix} = \begin{vmatrix} 17 & 2 & 6 \\ 13 & 2 & 3 \\ 11 & 2 & 6 \end{vmatrix} = 2\cdot 3\begin{vmatrix} 17 & 1 & 2 \\ 13 & 1 & 1 \\ 11 & 1 & 2 \end{vmatrix}$$

$$= 6\begin{vmatrix} 6 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 6\cdot 2\begin{vmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 12\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 12\cdot 3 = 36.$$

EXPLANATION. First we subtracted the first column from the second and third columns. This is equivalent to making two applications of Theorem VI, using $m=-1$ in each. Next, we have taken the factor 2 out of the second column of the resulting determinant, and the factor 3 out of its third column (Theorem IV). Next, we have subtracted 11 times the second column from the first column, and then taken out a factor 2 from the first column. Finally, we have subtracted 2 times the second column from the third. Note that the last determinant obtained has three zero elements, thus making its expansion relatively easy to calculate, giving 3. In general, the theorems of § 122 are to be thus employed to obtain one or more zero elements and correspondingly reduce the labor incident to the final expansion of a determinant. It is not to be expected, of course, that *all* the elements can be reduced to zero, or even all those in any one row or column, for this would imply that the determinant had the value zero, which in general would not be the case.

EXAMPLE 2.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c+a \\ 1 & b & c+a+b \\ 1 & c & a+b+c \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = (a+b+c)\cdot 0 = 0.$$

Note the application of Theorem III in the last step.

### EXERCISES

Evaluate the following determinants, employing as may be desired the theorems of § 122.

**1.** $\begin{vmatrix} 8 & 4 & 6 \\ 2 & -2 & 4 \\ 2 & 3 & 4 \end{vmatrix}$ **2.** $\begin{vmatrix} 20 & 15 & 25 \\ 17 & 12 & 22 \\ 19 & 20 & 16 \end{vmatrix}$ **3.** $\begin{vmatrix} 5 & 2 & 7 & 5 \\ 6 & 3 & 1 & 4 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 2 & 5 \end{vmatrix}$

**124. Minors.** If one row and one column of a determinant be erased, a new determinant of order one lower than the given determinant is obtained. This determinant is called a ***first minor*** of the given determinant. Similarly, by erasing two rows and two columns, we obtain a ***second minor;*** and so on.

Thus, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \tag{1}$$

by erasing the second row and third column, we obtain the first minor

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

This minor is said to correspond to the element $c_2$, since the row and column erased *both* contain this element. We may represent it, therefore, by $D_{c_2}$.

In general, to each element of (1) corresponds a first minor obtained by erasing the row and column in which that element stands. The minor of $a_1$ is represented by $D_{a_1}$, the minor of $a_2$ by $D_{a_2}$, etc.

Similar remarks evidently apply to a determinant of any order.

**125. Development According to Minors.** An examination of the expanded form of the typical determinant of the third order (see (2) of § 117), shows that it may be written if desired in the form

$$a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1),$$

or

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

which, by § 124, may be written in the form

$$a_1D_{a_1}-a_2D_{a_2}+a_3D_{a_3}.$$

Thus, we have the relation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1D_{a_1}-a_2D_{a_2}+a_3D_{a_3}. \tag{1}$$

As thus written, the determinant is said to be *developed according to the minors of its first column.*

In a similar way, we may show that the same determinant may be developed according to the minors of any given row or column, provided only that in forming the various products thus called for of elements into their minors, the following general rule be observed:

RULE. *The product of the element lying in the rth column and sth row multiplied by its minor is to be taken positively or negatively according as* $(r+s)$ *is even or odd.*

Thus, the determinant (1), when developed according to the elements of its second column, becomes

$$-b_1D_{b_1}+b_2D_{b_2}-b_3D_{b_3}.$$

Other illustrative forms of development for the same determinant are

$$-a_2D_{a_2}+b_2D_{b_2}-c_2D_{c_2},$$
$$a_3D_{a_3}-b_3D_{b_3}+c_3D_{c_3}.$$

Passing now to the typical determinant of the *fourth* order (see (1), § 121) it will be found, upon examining the terms of its expansion, that it may be developed according to the minors of any one of its rows or columns in the manner just described, and in fact a like statement may be verified for a determinant of any order whatever. For brevity, the details of the proof will be omitted.

Thus, the determinant (1) of § 121, when developed by minors according to the elements of its first column, becomes

$$a_1D_{a_1}-a_2D_{a_2}+a_3D_{a_3}-a_4D_{a_4}.$$

Here, of course, each of the minors, $D_{a_1}$, $D_{a_2}$, $D_{a_3}$, $D_{a_4}$, is a determinant of the *third* order.

Other illustrative forms of development for the same determinant are

$$-b_1D_{b_1}+b_2D_{b_2}-b_3D_{b_3}+b_4D_{b_4},$$
$$-a_2D_{a_2}+b_2D_{b_2}-c_2D_{c_2}+d_2D_{d_2},$$
$$c_1D_{c_1}-c_2D_{c_2}+c_3D_{c_3}-c_4D_{c_4}.$$

It is frequently advantageous to develop a determinant according to its minors, especially in case several of the elements in some column (or row) are equal to zero.

Example. Find the value of the determinant

$$\begin{vmatrix} 4 & 2 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 4 & 9 \end{vmatrix}.$$

Solution. First subtract the third row from the first (Theorem VI, § 122), thus obtaining as an equivalent determinant, and one having a number of zeros in its first row,the following

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 5 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 4 & 9 \end{vmatrix}.$$

Now develop by minors according to the elements of the first row.

$$1 \cdot \begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 2 \\ 6 & 4 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 2 \\ 5 & 4 & 9 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 3 & 5 \\ 3 & 2 & 2 \\ 5 & 6 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 5 & 6 & 4 \end{vmatrix}.$$

Of these four terms the last three vanish because of the factor 0 in each, so the result reduces to the determinant appearing in the first term. We may evaluate this third order determinant, as follows:

Multiplying the second column by 2 and subtracting it from both the first and last columns, this determinant takes the form

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{vmatrix}.$$

Developing this according to the elements of the second row, we have

$$-0 \cdot \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = -1+2=1.$$

Thus the value of the original determinant is 1.

## EXERCISES

Evaluate each of the following determinants by using the method of development by minors.

**1.** $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 7 & 9 \\ 1 & 2 & 1 & 4 \\ 5 & 6 & 3 & 2 \end{vmatrix}.$ **2.** $\begin{vmatrix} 2 & 1 & 3 & 7 \\ 1 & 2 & 4 & 6 \\ 1 & 0 & 2 & 0 \\ 2 & 3 & 5 & -4 \end{vmatrix}.$ **3.** $\begin{vmatrix} 5 & 2 & 7 & 5 \\ 6 & 3 & 1 & 4 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 2 & 5 \end{vmatrix}.$

**126. Cofactors.** If the minor of an element of a determinant be taken with its proper sign, as determined by the Rule of § 125, the result is called the ***cofactor*** of that element.

Thus, in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

the cofactor of $b_1$ is

$$-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix};$$

that of $b_2$ is

$$+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}.$$

It is customary to represent the cofactor of $a_1$ by $A_1$, the cofactor of $a_2$ by $A_2$, that of $a_3$ by $A_3$, that of $b_1$ by $B_1$, etc. By use of these cofactors the development of any given determinant is readily expressible in various forms, in accordance with the results of § 125.

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

may be expressed in any of the following forms.

$$a_1A_1+a_2A_2+a_3A_3, \qquad b_1B_1+b_2B_2+b_3B_3,$$

$$a_2A_2+b_2B_2+c_2C_2, \qquad c_1C_1+c_2C_2+c_3C_3, \text{ etc.}$$

In connection with cofactors, the following theorem is important.

THEOREM. *If the elements in any column be multiplied respectively by the cofactors of the corresponding elements in another column, the sum of the products is equal to zero.*

Thus, in the typical determinant of the third order (see above) we have

$$a_1B_1+a_2B_2+a_3B_3=0, \qquad b_1A_1+b_2A_2+b_3A_3=0,$$

$$a_1C_1+a_2C_2+a_3C_3=0, \qquad c_1B_1+c_2B_2+c_3B_3=0, \text{ etc.}$$

Proof. Consider the third order determinant (see above). For this we may write, as shown above,

$$(1)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = b_1B_1+b_2B_2+b_3B_3.$$

Now, no one of the cofactors $B_1$, $B_2$, $B_3$ contains any of the elements $b_1$, $b_2$, $b_3$. Hence, these cofactors are unaffected if in (1) we change $b_1$, $b_2$, $b_3$ to $a_1$, $a_2$, $a_3$. This gives

$$a_1B_1+a_2B_2+a_3B_3 = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}.$$

But this determinant is equal to zero by Theorem III, § 122, thus establishing as desired that $a_1B_1+a_2B_2+a_3B_3=0$.

The proof may evidently be extended to cover any case in any determinant.

**127. Simultaneous Equations.** It was shown in § 116 that a system of two simultaneous equations of the first degree between two unknown letters $x$, $y$ can be readily solved by means of determinants, and in § 118 a like fact was shown regarding the value of the three unknown letters $x$, $y$, $z$ pertaining to a similar system of three equations. The general formulas for such solutions are to be seen in (5) of § 116 and (3) of § 118, which should now be examined. We proceed to show that similar formulas exist also for the four values $x$, $y$, $z$, $w$ pertaining to a system of *four* equations of the first degree between these unknowns, and similarly for a system of five equations, etc. Suppose, then, that the system is one of four unknowns, namely,

$$\begin{aligned} a_1x+b_1y+c_1z+d_1w&=k_1,\\ a_2x+b_2y+c_2z+d_2w&=k_2,\\ a_3x+b_3y+c_3z+d_3w&=k_3,\\ a_4x+b_4y+c_4z+d_4w&=k_4. \end{aligned}$$

Consider the determinant

$$D=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and let $A_1, A_2, \cdots, B_1, B_2, \cdots$, etc., be its cofactors.

Multiplying the first equation by $A_1$, the second by $A_2$, the third by $A_3$ and the fourth by $A_4$ and adding, we have

$$(1)\quad \begin{aligned}&(a_1A_1+a_2A_2+a_3A_3+a_4A_4)x+\\ &(b_1A_1+b_2A_2+b_3A_3+b_4A_4)y+\\ &(c_1A_1+c_2A_2+c_3A_3+c_4A_4)z+\\ &(d_1A_1+d_2A_2+d_3A_3+d_4A_4)w=k_1A_1+k_2A_2+k_3A_3+k_4A_4.\end{aligned}$$

Here the coefficients of $y$, $z$ and $w$ each vanish by the theorem of § 126, so that (1) reduces to

$$(2)\quad (a_1A_1+a_2A_2+a_3A_3+a_4A_4)x=k_1A_1+k_2A_2+k_3A_3+k_4A_4.$$

The coefficient of $x$ in (2) is $D$; the right side is what $D$ becomes when the elements $a_1$, $a_2$, $a_3$, $a_4$ are respectively replaced by $k_1$, $k_2$, $k_3$, $k_4$. Solving (2) for $x$, we thus have

$$(3)\qquad x=\frac{\begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}}{D}.$$

In like manner, by multiplying the first of the given equations by $B_1$, the second by $B_2$, etc., and adding and applying the theorem of § 126, we obtain

$$(4)\qquad y=\frac{\begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}}{D}.$$

Likewise, the values of $z$ and $w$ are each expressible as the quotient of two determinants, the denominator in each instance being $D$ and the numerators being the determinants obtained from $D$ by replacing the elements of its *third* column and *fourth* column respectively by $k_1$, $k_2$, $k_3$, $k_4$.

Using for brevity the condensed form of notation explained in the Note at the close of § 121, the formulas for $x$, $y$, $z$, $w$ thus become respectively

$$x=\frac{|k_1b_2c_3d_4|}{|a_1b_2c_3d_4|},\ y=\frac{|a_1k_2c_3d_4|}{|a_1b_2c_3d_4|},\ z=\frac{|a_1b_2k_3d_4|}{|a_1b_2c_3d_4|},\ w=\frac{|a_1b_2c_3k_4|}{|a_1b_2c_3d_4|}.$$

These four formulas are seen to be analogous in formation to the three formulas obtained in § 118 where only three equations were under consideration.

Similar statements and results evidently apply to any set of simultaneous equations of the first degree containing as many unknown letters as equations.

NOTE. It is to be observed that in case the determinant $D$ which appears above has the value zero, the formulas (3), (4), etc. can no longer be used, since division by zero is not a permissible operation in mathematics. Such cases require special investigation and are considered in detail in higher algebra.

Similar remarks apply in general, and in particular to the systems already considered in §§ 116, 118.

**128. Elimination.** In all the systems of simultaneous equations thus far considered it was essential that the number of equations be the same as the number of unknown letters present. When this condition is not fulfilled, various possibilities may arise and, while space does not permit of their detailed study here, the single case in which the number of equations is one greater than the number of unknowns is particularly important and will therefore be briefly considered below.

Suppose, then, that three unknowns, $x$, $y$, $z$ are present and that these are to satisfy *four* equations of the first degree, which we shall write in the form

$$\begin{aligned} a_1x+b_1y+c_1z&=d_1,\\ a_2x+b_2y+c_2z&=d_2,\\ a_3x+b_3y+c_3z&=d_3,\\ a_4x+b_4y+c_4z&=d_4. \end{aligned}\tag{1}$$

Moreover, let us suppose that a certain three of these equations, say the first three, when treated as in § 127, may be solved for $x$, $y$, $z$. We have left to determine when these values of $x$, $y$, $z$ will satisfy also the fourth equation.

Now, noting the form of the solutions for $x$, $y$, $z$ in the first three equations (see (3), § 118) and placing them in the fourth equation, then clearing the latter of fractions it becomes (using the condensed notation explained in the Note at the close of § 121)

$$a_4\,|d_1b_2c_3|+b_4\,|a_1d_2c_3|+c_4\,|a_1b_2d_3|=d_4\,|a_1b_2c_3|.$$

Transposing all terms to the left side and noting that, by Theorem II, § 122, we may write $|d_1b_2c_3|=|b_1c_2d_3|$, $|a_1d_2c_3|=-|a_1c_2d_3|$, the last relation becomes (after multiplying through by $-1$)

$$-a_4\,|b_1c_2d_3|+b_4\,|a_1c_2d_3|-c_4\,|a_1b_2d_3|+d_4\,|a_1b_2c_3|=0.$$

But this relation is the same as

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=0,\tag{2}$$

as appears by expanding this determinant by minors according to the elements of its last row.

THEOREM. *In order that the system* (1) *may have a set of values $x$, $y$, $z$ that will satisfy it, it is necessary that condition* (2), *which relates only to the sixteen coefficients of the system, shall be satisfied.*

The determinant appearing in (2) is called the ***eliminant*** of the system (1). Thus, the theorem above may be stated briefly as follows. *In order that the system* (1) *may have a solution* $x$, $y$, $z$ *it is necessary that the eliminant of the system shall be equal to zero.*

A similar theorem may now be supplied for any system of linear equations containing one more equation than unknown quantities. The student is advised to do this for such a system of *five* equations.

### EXERCISES

Solve by determinants each of the following systems of equations.

**1.** $\begin{cases} 2x+3y-z+w=6, \\ x+y+z-2w=4, \\ 3x+2y-3z+w=-1, \\ x-y-z+3w=-1. \end{cases}$ **2.** $\begin{cases} 2x+3y-4z+w=0, \\ x-y+z-w=-2, \\ 7x+2y-3z+w=6, \\ 5x+8y-10z+3w=3. \end{cases}$

Form the eliminant for each of the following systems of equations and use it to tell (by the theorem of § 128) whether the system may have a solution. In cases where there may be a solution, proceed to determine it (if possible) by the methods of § 127.

**3.** $\begin{cases} 2x+3y=9, \\ 3x-y=8, \\ x+y=6. \end{cases}$ **4.** $\begin{cases} x+y=4, \\ 2x-y=5, \\ 3x-2y=7. \end{cases}$ **5.** $\begin{cases} 3x+2y+3z=17, \\ 2x+y+2z=10, \\ 5x+5y+z=29, \\ x+y+z=7. \end{cases}$

**6.** Find the value (or values) of $k$ for which the following system may have a solution.

$$\begin{cases} kx+3y=18, \\ x-7y=-8, \\ x-ky=2. \end{cases}$$

**7.** Eliminate $m$ from the system

(1) $$m^2x-mx^2=1,$$

(2) $$m+2x=2.$$

SOLUTION. FIRST METHOD. Solve (2) for $m$, giving $m=2-2x$, and place this value of $m$ in (1), giving as the desired result

$$(2-2x)^2x-(2-2x)x^2=1,$$

which upon reducing becomes

$$6x^3-10x^2+4x-1=0.$$

This equation in $x$ alone is, then, the result of eliminating $m$ from (1) and (2). It is an equation whose roots satisfy (1) and (2) *whatever the value of m.*

Second Method. Multiply (2) through by $m$, giving

$$m^2+2mx=2m. \tag{3}$$

Now, arrange (1), (2) and (3) in the forms

$$x\cdot m^2-x^2\cdot m=1, \tag{1}$$

$$0\cdot m^2+\ 1\cdot m=(2-2x), \tag{2}$$

$$1\cdot m^2+\ (2x-2)m=0. \tag{3}$$

Regarding this system as one of three linear equations between the *two* quantities $m^2$ and $m$, and applying the results of § 128, we obtain as the desired equation

$$\begin{vmatrix} x & -x^2 & 1 \\ 0 & 1 & (2-2x) \\ 1 & (2x-2) & 0 \end{vmatrix}=0.$$

Upon expanding this determinant it readily reduces to

$$6x^3-10x^2+4x-1=0$$

and this is seen to be the same result as obtained above by the first method.

In contrasting the two methods, it will be seen that the second does not depend upon solving either of the given equations for $m$, as did the first method. For this reason, the second method has a much wider range of applicability, as will be illustrated in the examples which follow. The second method illustrates what is known as *Sylvester's method of elimination*†.

**8.** Eliminate $m$ from the following system, using both the methods illustrated in Ex. 7 and noting that the result for either method is the same.

$$m^2x-2mx^2+1=0,$$
$$m+x^2-3mx=0.$$

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† For details, see for example Burnside and Panton's *Theory of Equations* (Longmans, Green and Co.), Chapter on Elimination.

**9.** Write as a determinant the result of eliminating $k$ from the system

$$kx-k^2x^2+k=1,$$
$$2k^2x^3+kx^2-k=2.$$

[HINT. Multiply each equation through by $k$ and consider the resulting equations combined with the original ones.]

**10.** Find the condition (in the form of a determinant) that the two equations

$$a_1x^2+b_1x+c_1=0,$$
$$a_2x^2+b_2x+c_2=0,$$

may have a common root.

[HINT. The result of eliminating $x$, where $x$ is regarded as the common root, will express the desired condition.]

**11.** Find the condition (in the form of a determinant) that the two equations

$$ax^2+bx+c=0$$
$$x^3+qx+r=0,$$

may have a common root.

**12.** Determine the value (or values) of $k$ for which the following two equations may have a common root.

$$2x^2-7x+3=0,$$
$$x^2+kx+15=0.$$

# CHAPTER XV

## COMPLEX NUMBERS

**129. Introduction.** In § 10 it was pointed out that every complex number is of the type form $a+bi$, where $a$ and $b$ are real numbers and where $i$ represents the quantity $\sqrt{-1}$. If in particular $b=0$, the complex number reduces to an ordinary real number, $a$, while if $a=0$ ($b$ being different from 0), the complex number takes the form $bi$, which is known as a pure imaginary. Complex numbers commonly enter into the roots of quadratic equations and other equations of higher degree than the first, as shown in earlier chapters. They have also many other important applications. For this reason we shall now consider their properties in somewhat further detail.

**130. Complex Numbers Considered Graphically.** For convenience, let us for the moment represent the typical complex number by $x+yi$, where $x$ and $y$ are any real numbers and where $i=\sqrt{-1}$. If, then, the values of $x$ and $y$ are given, they may be regarded respectively as the abscissa and the ordinate of a certain point $P$ in the manner described in § 6, in which case $P$ may be taken as the geometrical representation of the complex number $x+yi$. Conversely, to every point $P$ in the plane there evidently corresponds a definite complex number $x+yi$.

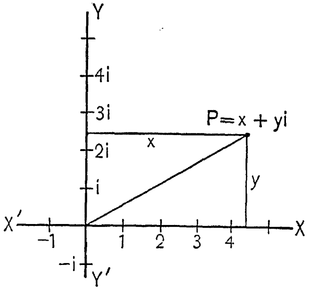

FIG. 58

Whenever complex numbers are thus represented, the axis $X'X$ is known as the ***axis of reals***, while the axis $Y'Y$ is called the ***axis of pure imaginaries***, the appropriateness of these names being at once apparent. The axis $Y'Y$ is thus regarded as a line along which all pure imaginary numbers may be represented, the units along this axis being therefore taken as $i$, $2i$, $3i$, $4i$, etc. as in Fig. 58. The $X'X$ axis, on the other hand, furnishes a representation of all real numbers.

It may be remarked also that the plane determined by the axis of reals and the axis of pure imaginaries is frequently referred to in higher mathematics as the ***complex plane***. As thus defined, the complex plane evidently presents a geometric picture of all the numbers that enter into algebra.

**131. Definitions.** Two complex numbers $a+bi$ and $c+di$ are said to be ***equal*** if, and only if, $a=c$ and $b=d$; that is, if the real parts $a$ and $c$ are equal to each other, and the imaginary parts, $b$ and $d$, are likewise equal to each other.

Two complex numbers which differ only in the signs of their imaginary parts are called ***conjugate complex numbers.***

Thus, $2+3i$, $2-3i$ are conjugate numbers; likewise $-\frac{1}{2}+\frac{1}{3}i$, $-\frac{1}{2}-\frac{1}{3}i$.

It will be observed that the type form of every pair of conjugate numbers is $a+bi$, $a-bi$ and that two such numbers, when interpreted geometrically by the method of § 130, give rise to two points $P_1$ and $P_2$, which are symmetrically situated with respect to the axis of reals; that is, the one lies above this axis the same distance that the other lies below it, while the line joining the two is perpendicular to this same axis.

**132. Addition and Subtraction of Two Complex Numbers.** Two complex numbers $a+bi$ and $c+di$ are added or subtracted in accordance with following formulas:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

These formulas, when expressed in words, may be stated as follows:

*To add (or subtract) two complex numbers, add (or subtract) the real parts separately and likewise add (or subtract) the imaginary parts separately. The resulting complex number is the sum (or difference) desired.*

It is now to be observed that any two complex numbers may be readily added graphically. Thus, in Fig. 59 let the two numbers, which we will call $a+bi$ and $c+di$, be represented respectively by the points $P_1$ and $P_2$. Connect $P_1$ and $P_2$ each with the origin $O$ and then complete the parallelogram which has $OP_1$ and $OP_2$ as two of its adjacent sides. The vertex $P$ thus determined represents the sum of the given numbers $a+bi$ and $c+di$, for we may evidently write

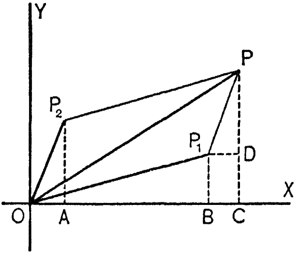

FIG. 59

$$OC=OB+BC=OB+OA=a+c,$$
$$CP=CD+DP=BP_1+AP_2=b+d,$$

so that $P$ corresponds to the complex number $(a+c)+(b+d)i$, and this, by the formulas above, is the sum of $a+bi$ and $c+di$.†

---

† It should be noted that this figure is the same as that used to represent the composition of forces in physics.

To subtract one complex number from another graphically, as $c+di$ from $a+bi$, we have but to represent the points corresponding to the numbers $-c-di$, $a+bi$ and proceed as in addition.

**Note.** The processes discussed in this section illustrate what is known in mathematics as the addition and subtraction of vectors, a ***vector*** being defined as any magnitude which depends for its meaning upon its direction as well as its length, as is the case with $OP_1$ or $OP_2$ in Fig. 59. The general study of vectors and the operations that can be performed upon them is known as ***vector analysis***. The subject is of especial importance in the general applications of mathematics to physics and electrical engineering.

## EXERCISES

Represent by a drawing on coordinate paper each of the following numbers.

**1.** $1+3i$.
**2.** $-2+2i$.
**3.** $1-i$.
**4.** $5$.
**5.** $5i$.
**6.** $-4+0i$.
**7.** $1\frac{1}{2}-\frac{1}{4}i$.
**8.** $-.7-1.3i$.
**9.** $-5\frac{1}{2}i$.
**10.** $\sqrt{2}+\frac{2}{3}i$.
**11.** $\frac{2}{3}+\sqrt{2}i$.
**12.** $1.3-\sqrt{5}i$.

State which of the following are pairs of conjugate numbers.

**13.** $\begin{cases} 3+2i, \\ 4-2i. \end{cases}$
**14.** $\begin{cases} 2-i, \\ 2+i. \end{cases}$
**15.** $\begin{cases} 2i, \\ -3i. \end{cases}$
**16.** $\begin{cases} \frac{2}{3}i, \\ -\frac{2}{3}i. \end{cases}$

By use of the definition of equal complex numbers, find what must be the value of $x$ and $y$ in each of the following equations.

**17.** $x+y-(x-y)i=3+i$.
**18.** $3x+5y+(2x-y)i=2-3i$.
**19.** $x^2+y^2+(x-3y)i+2yi=2+2i$.
**20.** $2x-2y-2xi+y+xi+xyi=5-4i$.

Perform the following operations both algebraically and geometrically.

**21.** $(2+i)+(3-2i)$.
**22.** $(3+2i)+(-2+3i)$.
**23.** $(-5-i)+(-2+4i)$.
**24.** $(1+2i)+(-2-i)+(3-4i)$.
**25.** $(3+2i)-(2+3i)$.
**26.** $(-4+5i)-(3-i)$.
**27.** $(5+i)-(3-5i)$.
**28.** $(3-4i)+(2+i)-(5-6i)$.

**133. Complex Numbers in Polar Form.** Let the point $P$ corresponding to the complex number $x+yi$ be joined to the origin $O$ by the line $OP$. Then the length of $OP$, which is generally denoted by $r$, is called the ***absolute value,*** or the ***modulus***, of the complex number $x+yi$; also, the angle made by $OP$ with the positive direction of the X-axis is called the ***angle***, or the ***argument***, of the complex number $x+yi$, and is commonly represented by the letter $\theta$. Moreover, by elementary trigonometry, we evidently have

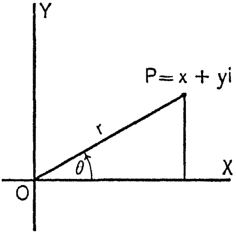

FIG. 60

$$(1) \qquad x=r\cos\theta,\ \ y=r\sin\theta.$$

Hence we may write

$$x+yi=x+iy=r\cos\theta+ir\sin\theta,$$

or

$$\boldsymbol{x+yi=r(\cos\theta+i\sin\theta).}$$

As thus written, the complex number $x+yi$ is said to be expressed in ***polar form.*** Moreover, the line-segment $OP$, when regarded both as to its length and its direction, may be taken as the geometric representation of the same number, namely, of the number $x+yi$. We shall make frequent use of these conceptions in what follows.

NOTE. It is to be observed that as different complex numbers $x+yi$ are selected in the plane, the corresponding values of the modulus $r$ will range from 0 to indefinitely large values, while the values of the argument $\theta$ will range from 0° to 360° (or 0 to $2\pi$ radians).

By means of relations (1) the value of $r$ and of $\theta$ belonging to any given complex number $x+yi$ may be readily determined, as illustrated in the following example.

Example. Determine the absolute value and the angle of the complex number $2-2i$.

Solution. From equations (1) we have

$$x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2(\sin^2\theta+\cos^2\theta)=r^2.$$

Hence, in the present instance

$$2^2+(-2)^2=r^2.$$

Therefore,

$$r=\sqrt{2^2+(-2)^2}=\sqrt{8}=2\sqrt{2}.$$

Using this value of $r$ in (1), we now obtain

$$\cos\theta=\frac{x}{r}=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}$$

and

$$\sin\theta=\frac{y}{r}=\frac{-2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}.$$

Therefore, by elementary trigonometry, $\theta=315°$.

The desired values of the absolute value and of the angle of $2-2i$ are therefore

$$r=2\sqrt{2},\ \theta=315°.$$

The student should now represent this number graphically.

**134. The Multiplication of Complex Numbers.** Any two complex numbers $a+bi$ and $c+di$ may be multiplied together, giving another complex number as the product. Thus, by ordinary multiplication we have

$$(a+bi)(c+di)=ac+ibc+iad+i^2bd.$$

But $i^2=(\sqrt{-1})^2=-1$, so that this equation becomes

$$(a+bi)(c+di)=(ac-bd)+(bc+ad)i.$$

Thus the product is the complex number whose real part is $(ac-bd)$ and whose pure imaginary part is $(bc+ad)i$.

It is important to note the nature of the product of two complex numbers when each is given in polar form. Thus, if the two given numbers are $r_1(\cos\theta_1+i\sin\theta_1)$ and $r_2(\cos\theta_2+i\sin\theta_2)$, we obtain by ordinary multiplication, after recalling that $i^2=-1$,

$$r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)$$
$$=r_1r_2[\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)].$$

But by familiar formulas in elementary trigonometry we have

$$\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2=\cos(\theta_1+\theta_2)$$

and

$$\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2=\sin(\theta_1+\theta_2).$$

Thus, the result at the top of this page may be written in the form

$$\boldsymbol{r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)}$$
$$\boldsymbol{=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)].}$$

Expressed in words, this formula may be summarized as follows:

*The absolute value of the product of two complex numbers is equal to the product of their absolute values and the angle is equal to the sum of their angles.*

This result leads at once to the following geometric method for multiplying any two complex numbers. Draw through the origin $O$ the straight line which makes the angle $\theta_1+\theta_2$ with the positive direction of the real axis, and lay off on this line from $O$ a segment of length $r_1r_2$. The directed line-segment $OP$ thus determined corresponds to the product desired.

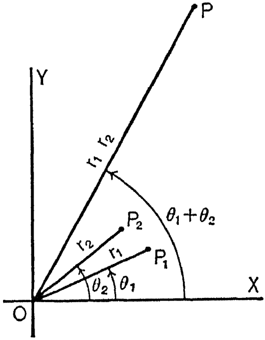

FIG. 61

For example,

$$5(\cos 10^\circ+i\sin 10^\circ)\cdot 3(\cos 20^\circ+i\sin 20^\circ)$$
$$=15[\cos(10^\circ+20^\circ)+i\sin(10^\circ+20^\circ)]$$
$$=15[\cos 30^\circ+i\sin 30^\circ]=15\left[\frac{\sqrt{3}}{2}+i\tfrac{1}{2}\right]=\tfrac{15}{2}(\sqrt{3}+i).$$

**135. The Division of Complex Numbers.** Any complex number $a+bi$ may be divided by any other complex number $c+di$ (it being assumed that not both $c$ and $d$ are equal to zero), the quotient being another complex number. Thus, we may write (by multiplying both numerator and denominator by $c-di$ and simplifying the result)

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{(ac+bd)-i(ad-bc)}{c^2+d^2}$$

$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i.$$

The quotient is therefore the complex number whose real part is $(ac+bd)/(c^2+d^2)$ and whose pure imaginary part is $i(bc-ad)/(c^2+d^2)$.

It is important to note the nature of the quotient of two complex numbers when each is given in polar form. Thus, if the two numbers are $r_1(\cos\theta_1+i\sin\theta_1)$ and $r_2(\cos\theta_2+i\sin\theta_2)$, we may write

$$\frac{r_1(\cos\theta_1+\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)}{r_2^2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$$=\frac{r_1r_2[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]}{r_2^2(\cos^2\theta_2+\sin^2\theta_2)},$$

as appears by first multiplying both numerator and denominator by $r_2(\cos\theta_2-i\sin\theta_2)$ and then simplifying the result by means of the familiar trigonometric formulas

$$\sin(\theta_1-\theta_2)=\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2,$$

$$\cos(\theta_1-\theta_2)=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2.$$

Thus we arrive at the formula

$$\boldsymbol{\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)].}$$

Expressed in words, this formula may be summarized as follows:

*The absolute value of the quotient of two complex numbers is equal to the quotient of their absolute values and the angle is equal to the difference of their angles.*

This result leads at once to the following geometric method for dividing one complex number, as $r_1(\cos\theta_1+i\sin\theta_1)$ by any other, as $r_2(\cos\theta_2+i\sin\theta_2)$. Draw through the origin $O$ the straight line which makes the angle $\theta=\theta_1-\theta_2$ with the positive direction of the real axis, and lay off on this line from $O$ a segment of length $r=r_1/r_2$. The directed line-segment $OP$ thus determined corresponds to the quotient desired.

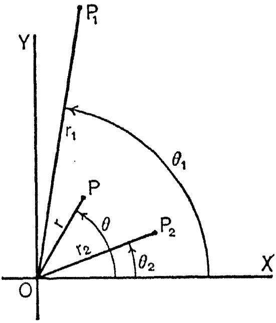

FIG. 62

For example,

$$\frac{10\,(\cos 45^\circ+i\sin 45^\circ)}{2\,(\cos 30^\circ+i\sin 30^\circ)}=5[\cos(45^\circ-30^\circ)+i\sin(45^\circ-30^\circ)]$$
$$=5[\cos 15^\circ+i\sin 15^\circ].$$

## EXERCISES

In each of the following exercises determine the indicated product both analytically and graphically.

1. $(1+i)(-1+2i)$.

SOLUTION. Considered analytically, the product is found by direct multiplication as follows:

$$(1+i)(-1+2i)=-1+2i-i+2i^2=-3+i.$$

To work the problem graphically, we use the result stated in § 134. Thus, we first draw the line-segments $OP_1$, $OP_2$ corresponding respectively to the numbers $1+i$, $-1+2i$. We then draw a line through

$O$ making an angle with the positive real axis which is equal to the sum of the angles $XOP_1$, $XOP_2$. Upon this line we lay off a distance $OP$ equal to the product of the two lengths $OP_1$, $OP_2$.† The directed line-segment $OP$ thus obtained represents the product desired.

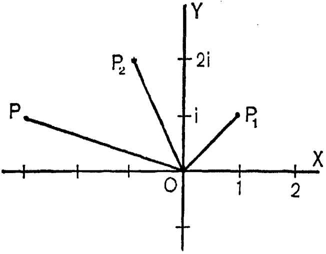

FIG. 63

**2.** $(3+i)(2-i)$.

**3.** $(1+2i)(1-3i)$.

**4.** $(-2+3i)(3-2i)$.

**5.** $(4-2i)(\frac{2}{3}+3i)$.

**6.** $(1+i)^2$.

**7.** $(2+3i)^2$.

**8.** $(0+2i)(1-3i)$.

**9.** $(0+2i)(0-2i)$.

**10.** $(1+i)(1+2i)(1+3i)$.

**11.** $(3+2i)(2-2i)(1+3i)$.

In each of the following exercises determine the indicated quotient both analytically and graphically.

**12.** $(-1+3i)\div(1+i)$.

SOLUTION. Considered analytically, the quotient is found by multiplying both the numerator and the denominator of the given fraction by $1-i$. Thus

$$\frac{-1+3i}{1+i}=\frac{(-1+3i)(1-i)}{(1+i)(1-i)}=\frac{-1+i+3i+3}{1+1}=\frac{2+4i}{2}=1+2i.$$

---

† The manner in which a line-segment $OP$ may be obtained by construction whose length shall be equal to the product of two given line-segments is shown in elementary geometry, and should be here employed. See, for example, Ford and Ammerman's Plane and Solid Geometry, Revised Edition (Macmillan), Example 21, page 176; also Problem 2, page 154.

To solve the problem graphically, we use the result stated in § 135. First draw $OP_1$, $OP_2$ corresponding respectively to $-1+3i$ and $1+i$. Then draw $OP$ so that the angle $XOP$ shall be equal to the angle $XOP_1$ minus the angle $XOP_2$, and lay off $OP$ equal to the quotient of the length of $OP_1$ divided by the length of $OP_2$.† The resulting directed line-segment $OP$ represents the quotient desired.

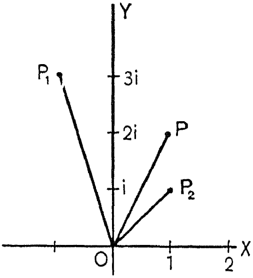

FIG. 64

**13.** $(2-3i)\div(1-2i)$.

**14.** $(3+i)\div(2-i)$.

**15.** $(-2+3i)\div(3-2i)$.

**16.** $(4-2i)\div(\frac{2}{3}+3i)$.

**17.** $(1+2i)\div(1-3i)$.

**18.** $(1+2i)\div(1-2i)$.

**19.** $(0+2i)\div(1-3i)$.

**20.** $(0+2i)\div(0-2i)$.

**21.** $(-1+i)\div(2-0i)$.

**22.** $\frac{(2+i)(3+i)}{1+2i}$.

In each of the following quotients determine first the value of the numerator and denominator separately, doing the work both analytically and graphically, then determine both analytically and graphically the value of the quotient itself.

**23.** $\frac{(2+3i)(1-i)}{(3-i)(2+i)}$.

**24.** $\frac{(2-i)(3+2i)(-1+i)}{(3-i)(2+i)(1+i)}$.

**136. De Moivre's Theorem.** Letting $r(\cos\theta+i\sin\theta)$ be any complex number expressed in polar form, we have

$$[r(\cos\theta+i\sin\theta)]^2=r^2(\cos\theta+i\sin\theta)^2$$
$$=r^2[(\cos^2\theta-\sin^2\theta)+i\,2\cos\theta\sin\theta].$$

But by trigonometry we have $\cos^2\theta-\sin^2\theta=\cos 2\theta$, and also $2\cos\theta\sin\theta=\sin 2\theta$. Hence we may write

$$[r(\cos\theta+i\sin\theta)]^2=r^2(\cos 2\theta+i\sin 2\theta).$$

---

† For the method of geometric division by construction, see for example Ford and Ammerman's Plane and Solid Geometry (Revised Edition), Example 19, page 175.

If we now multiply each member of this equation by $r(\cos\theta+i\sin\theta)$, we obtain

$$[r(\cos\theta+i\sin\theta)]^3$$
$$=r^3[(\cos 2\theta\cos\theta-\sin 2\theta\sin\theta)+i(\cos 2\theta\sin\theta+\sin 2\theta\cos\theta)]$$

and this, by trigonometry, reduces to the form

$$[r(\cos\theta+i\sin\theta)]^3=r^3(\cos 3\theta+i\sin 3\theta).$$

Similarly, from this equation we may now derive the relation

$$[r(\cos\theta+i\sin\theta)]^4=r^4(\cos 4\theta+i\sin 4\theta).$$

Proceeding in this manner, we may easily establish by mathematical induction (Chapter IX) the general formula

$$[r(\cos\theta+i\sin\theta)]^n=r^n(\cos n\theta+i\sin n\theta),$$

$n$ being any positive integer. This formula is known as **De Moivre's Theorem.** It may be shown that it holds true not only when $n$ is a positive integer, but when it is any real number whatever. However, for simplicity we shall here omit the proof of the theorem for these more complicated cases.

For example, we may write

$$[8(\cos 30^\circ+i\sin 30^\circ)]^{\frac{2}{3}}=8^{\frac{2}{3}}(\cos\tfrac{2}{3}\cdot 30^\circ+i\sin\tfrac{2}{3}\cdot 30^\circ)$$
$$=4(\cos 20^\circ+i\sin 20^\circ).$$

**137. The Roots of Complex Numbers.** By trigonometry we may write

$$r(\cos\theta+i\sin\theta)=r\{\cos(\theta+m\,360^\circ)+i\sin(\theta+m\,360^\circ)\},$$

where $m$ is any positive integer. If we raise each member of this equation to the power $1/n$, where $n$ is any positive integer, and subsequently rewrite the second member of the result in the form furnished for it by De Moivre's theorem, we obtain the following formula for obtaining the $n$th root of any complex number:

$$[r(\cos\theta+i\sin\theta)]^{\frac{1}{n}}=r^{\frac{1}{n}}\left[\cos\frac{\theta+m\,360^\circ}{n}+i\sin\frac{\theta+m\,360^\circ}{n}\right],$$

$m$ being, as stated before, any positive integer (0 included). If in this formula we allow $m$ to take on in succession the values 0, 1, 2, 3, 4, ⋯, $n-1$, we thereby obtain $n$ distinct numbers each being an $n$th root of the given complex number $r(\cos\theta+i\sin\theta)$. If $m$ proceeds to take still larger values, however, no new roots are obtained, since the complex numbers thus resulting readily reduce by trigonometry to those already obtained. We may then state the following result:

*Any number (real or complex) has $n$ distinct $n$th roots, and no more.*

## EXERCISES

By means of De Moivre's theorem, find each of the following indicated powers and check your result by actual multiplication.

**1.** $(2-2i)^3$.

Solution. Writing $(2-2i)^3$ in polar form (see example in § 133), we have

$$2-2i=2\sqrt{2}(\cos 315^\circ+i\sin 315^\circ).$$

Hence, applying De Moivre's theorem,

$$\begin{aligned}(2-2i)^3&=(2\sqrt{2})^3(\cos 3\cdot 315^\circ+i\sin 3\cdot 315^\circ)\\&=16\sqrt{2}(\cos 945^\circ+i\sin 945^\circ)\\&=16\sqrt{2}(\cos 225^\circ+i\sin 225^\circ)\\&=16\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=-16(1+i).\end{aligned}$$

Check.

$$(2-2i)^3=(2-2i)^2(2-2i)=(-8i)(2-2i)=-16i-16=-16(1+i).$$

**2.** $(2-2i)^2$.

**3.** $(2-2i)^4$.

**4.** $(3+\sqrt{3}i)^2$.

**5.** $(3+\sqrt{3}i)^3$.

**6.** $(-\frac{1}{2}-\frac{1}{2}\sqrt{3}i)^3$.

**7.** $(3+\sqrt{3}i)^4$.

**8.** $(2+2i)^5$.

**9.** $(1+i)^{10}$.

By means of the formula of § 137, find the values pertaining to each of the following indicated roots. Show also the geometric significance of your results.

**10.** $\sqrt[3]{2-2i}$.

Solution. Using the polar form for $2-2i$ as given in Ex. 1, the formula of § 137 becomes in the present instance

$$(2-2i)^{\frac{1}{3}}=(2\sqrt{2})^{\frac{1}{3}}\left[\cos\frac{315^\circ+m\,360^\circ}{3}+i\sin\frac{315^\circ+m\,360^\circ}{3}\right].$$

By allowing $m$ to take in succession the values 0, 1, 2 we thus obtain the following three roots:

$$(2\sqrt{2})^{\frac{1}{3}}\left[\cos\frac{315^\circ}{3}+i\sin\frac{315^\circ}{3}\right]=\sqrt{2}(\cos 105^\circ+i\sin 105^\circ),$$
$$(2\sqrt{2})^{\frac{1}{3}}\left[\cos\frac{675^\circ}{3}+i\sin\frac{675^\circ}{3}\right]=\sqrt{2}(\cos 225^\circ+i\sin 225^\circ),$$
$$(2\sqrt{2})^{\frac{1}{3}}\left[\cos\frac{1035^\circ}{3}+i\sin\frac{1035^\circ}{3}\right]=\sqrt{2}(\cos 345^\circ+i\sin 345^\circ).$$

It is to be observed that these three roots, considered geometrically, give rise to three points $P_1$, $P_2$, $P_3$, which lie at equal distances apart upon a circle of radius $\sqrt{2}$ whose center is at the origin. A similar symmetric property is characteristic of the location of the $n$ distinct $n$th roots of any given number.

**11.** $\sqrt[5]{2-2i}$.
**12.** $\sqrt[3]{2+2i}$.
**13.** $\sqrt[5]{2+2i}$.
**14.** $\sqrt{3+\sqrt{3}i}$.
**15.** $\sqrt{64(\cos 120^\circ+i\sin 120^\circ)}$.
**16.** $\sqrt[3]{8(\cos 60^\circ+i\sin 60^\circ)}$.
**17.** $\sqrt[5]{64(\cos 100^\circ+i\sin 100^\circ)}$.
**18.** $\sqrt[3]{-\frac{1}{2}-\frac{1}{2}\sqrt{3}i}$.
**19.** $\sqrt[3]{i}$. [Hint. Write in the form $\sqrt[3]{0+i}$.]
**20.** $\sqrt[5]{i}$. **21.** $\sqrt{16}$. **22.** $\sqrt[3]{1}$. **23.** $\sqrt[4]{1}$. **24.** $\sqrt[4]{16i}$.

Find all the roots of each of the following equations and represent them graphically.

**25.** $x^3-1=0$.

[Hint. The given equation is equivalent to $x=\sqrt[3]{1+0\cdot i}$.]

**26.** $x^3+1=0$.
**27.** $x^4-16=0$.
**28.** $x^5-32=0$,
**29.** $x^6-1=0$.
**30.** $x^4+1=0$.
**31.** $x^5+1=0$.

# ELEMENTARY EXERCISES

## A LIST OF EXERCISES FOR REVIEW OF HIGH SCHOOL ALGEBRA

### I. Addition and Subtraction

**1.** Add $7a+6b-3c$ and $4a-7b+4c$.

**2.** Add $2x+3y-2xy$, $7xy-4x-9y$, and $7x-5xy-4y$.

**3.** Add $x-8-7x^2+15x^3$, $4+14x^3-11x-x^2$, and $5x^2-9x^3+10x-12$.

**4.** Add $4(m+n)-3(q-r)$ and $4(m+n)+6(q-r)$.

**5.** Add $10(a+b)-11(b+c)$, $3(a+b)-5(c+d)$, and $3(b+c)-4(c+d)$.

**6.** Add $2mx+3nx-4qx$, $nx+2qx-ry$, and $py-qz+3w$.

**7.** Subtract $3x-2y+z$ from $5x-y+z$.

**8.** Subtract $4x^3-8-13x^2+15x$ from $6x^2+19x^3-4+12x$.

**9.** From $13a+5b-4c$ subtract $8a+9b+10c$.

**10.** From $2a+3c+d$ subtract $a-b+c$.

**11.** From $3x^2+7x+10$ subtract $-x^2-x-6$.

**12.** Subtract $1-a+a^2-a^3$ from $1-a^3$.

**13.** From the sum of $x^2-4xy+y^2$ and $6x^2-2xy+3y^2$ subtract $3x^2-5xy+7y^2$. Do it all in one operation if you can.

**14.** From the sum of $2s+8t-4w$ and $3s-6t+2w$ take the sum of $8s+9t+6w$ and $4s-7t-4w$.

Remove the parentheses and combine terms in each of the following expressions.

**15.** $a-(2a+4)-(5a+10)$.

**16.** $6x+(5x-\{2x+1\})$.

[Hint. Remove the innermost group sign first.]

**17.** $x-\{x-(x-3x)\}$.

**18.** $20z-[(2z+7w)-(3z+5w)]$.

## II. Multiplication

**19.** Multiply $x-1-3x^3+2x^4$ by $2+x$.

[Hint. First arrange the polynomials in *ascending* powers of $x$.]

Carry out each of the following indicated multiplications, first rearranging where desirable.

**20.** $(3a^2-2a-1)(a-1)$. **24.** $(2+3x^2-x+x^3)(x^2-2x+4)$.
**21.** $(x-3+x^2)(2+x)$. **25.** $(x^2+2xy+y^2)(x+y)$.
**22.** $(3a^2+2a-4)(5-a)$. **26.** $(a^2-ab+b^2)(a+b)$.
**23.** $(5n-4+6n^2)(8+n^2-4n)$. **27.** $(m^2-mn+n^2)(m^2+mn+n^2)$.
**28.** $(8r^2-2s^2+4rs)(2rs+3s^2+4r^2)$.
**29.** $(A^2+B^2-2AB)(A^2+B^2+2AB)$.
**30.** $(x^2y^2+xy+1)(1-xy+x^2y^2)$.

## III. Division

Perform each of the following divisions, and check your answer:

**31.** $a^6\div a^2$. **32.** $(3g)^9\div(3g)^2$. **33.** $(3ab)^8\div(3ab)^4$.
**34.** $(\frac{1}{2}p^2q)^7\div(\frac{1}{2}p^2q)^3$. **37.** $\frac{4}{3}\pi r^3\div 2\pi r$.
**35.** $-16x^2y^3z^2\div 4xy^2z$. **38.** $3ab(a+b)^2\div[-2(a+b)]$.
**36.** $4a^4b^2c^3\div 20a^2b^2c$. **39.** $(9m^3np+18mn^3p)\div 3mn$.
**40.** $(6x^2yz+12xy^2z-24xyz^2)\div(-3xyz)$.

In each of the following divisions, find the quotient, also the remainder if there is one. Check your answer for each.

**41.** $(3x^2-2x-1)\div(x-1)$. **42.** $(15x^2+x-2)\div(3x-1)$.
**43.** $(4y^3+2y^2-1)\div(2y-1)$. **44.** $(6x^3-7x^2+1)\div(2x-1)$.
**45.** $(x^3+x^2-x+2)\div(x^2-x+1)$. **46.** $(2x^2+3x+1)\div(x+2)$.
**47.** $(x^4-3x^3+x^2+2x-1)\div(x^2-x-2)$.
**48.** $(a^3-2a^2b+2ab^2-b^3)\div(a-b)$. **49.** $(x^4-y^4)\div(x+y)$.

## IV. Factoring

Factor each of the following expressions:

**50.** (*a*) $x^2+2x$. (*b*) $x^2y+xy^2$. (*c*) $8a^2+24a$. (*d*) $ab^2+b^3-b^2c$.
(*e*) $25c^2dx^3+35c^3d^2x^4-55c^2d^2x^5$. (*f*) $m(3x-1)-n(3x-1)$.
(*g*) $m(a+b-c)+n(a+b-c)-q(a+b-c)$.
(*h*) $pq-px-rq+rx$.
(*i*) $y^2-4y+xy-4x$. (*j*) $3x^3-15x+10y-2x^2y$.

**51.** (*a*) $81-x^2$.
(*b*) $a^2-b^2c^2$.
(*c*) $144x^2-4$.
(*d*) $\frac{1}{4}x^2y^2-36$.
(*e*) $9b^2-(a-x)^2$.
(*f*) $49a^2-(5a-4b)^2$.
(*g*) $(2x+5)^2-(5x-3)^2$.
(*h*) $(x+x^2)^2-(2x+2)^2$.

**52.** (*a*) $x^2+6x+8$.
(*b*) $x^2-6x+8$.
(*c*) $y^2+y-42$.
(*d*) $x^2-13x-48$.
(*e*) $x^2+xy-56y^2$.
(*f*) $12+7a+a^2$.
(*g*) $y^2-3ny-28n^2$.
(*h*) $a^2x^2+ax-12$.

**53.** Test each of the following expressions to see whether it is a trinomial square, and if so, factor it.

(*a*) $x^2-8x+16$.
(*b*) $x^2-12x+36$.
(*c*) $4x^2+6x+1$.
(*d*) $81x^2+18x+1$.
(*e*) $81-72r+16r^2$.
(*f*) $16y^2-24y+9$.
(*g*) $4x^2y^2-20xy+25$.
(*h*) $x^2+2x(x-y)+(x-y)^2$.
(*i*) $16-24(a-b)+9(a-b)^2$.
(*j*) $(a+b)^2-2(a+b)(b+c)+(b+c)^2$.

Factor *completely* each of the following expressions:

**54.** $x^3-x^2-x+1$.

[HINT. Write in the form $x^2(x-1)-(x-1)$.]

**55.** $2x^3-8x^2y+8xy^2$.

[HINT. Write in the form $2x(x^2-4xy+4y^2)$.]

**56.** $x^2+3ax-3a-x$.

**57.** $a^3+2a^2-4a-8$.

**58.** $x^4-13x^2+36$.

**59.** $x^4+y^4-2x^2y^2$.

**60.** $x^4-(2x-1)^2$.

**61.** $1-a^2-b^2+2ab$.

[HINT. $1-a^2-b^2+2ab$ $=1-(a-b)^2$.]

**62.** $m^2-4mn+4n^2-16$.

**63.** $2xy-x^2-y^2+1$.

**64.** $1+9c^2+6c$.

**65.** $(x^2-1)^2+(2x+3)(x-1)^2$.

**66.** $3x^3-3x+4x^4-4x^2$.

**67.** $1-a^2b^2-x^2y^2+2abxy$.

**68.** $(x^2-y^2)^2-(x^2-xy)^2$.

**69.** $x^3+y^3+x^2-y^2$.

**70.** $(x+1)^3-x^6$.

**71.** $(1-2x)^2-x^4$.

**72.** $6x^2+7x-3$.

**73.** $20x^2-6x-2$.

**74.** $8x^2-18xy-5y^2$.

**75.** $x^4-16$.

**76.** $11x^2+34x+3$.

**77.** $25s^4-25s^2t^2+4t^4$.

**78.** $9x^4-30x^2a+25a^2$.

**79.** $a^2+b^2+c^2-2ab+2ac-2bc$.

**80.** $(a-x^2)^2-(b-y^2)^2$.

**81.** $x^3+y^3+x+y$.

**82.** $x^2-9y^2+x+3y$.

**83.** $a^3-2a^2b+4ab^2-8b^3$.

**84.** $16x^4-y^2-6x^2y$.

**85.** $x^4-47x^2y^2+90y^4$.

## V. Fractions

Reduce to lowest terms:

**86.** $\frac{a^2-2a-15}{a^2+2a-35}.$

**87.** $\frac{4x^2-9}{4x^2-12x+9}.$

**88.** $\frac{x^3-3x^2-4x}{x^3-8x^2+16x}.$

**89.** $\frac{rs+ry-bs-by}{cs+cy-ks-ky}.$

**90.** $\frac{a(a+2b)^4(a^3+2a^2b+ab^2)}{b(a^2-4b^2)^2(a^5-2a^3b^2+ab^4)}.$

Reduce each of the following expressions to a single fraction:

**91.** $a+\frac{a^2-ab}{a}.$

**92.** $x-\frac{a-c-s}{d}.$

**93.** $1+m+\frac{m^2}{1+m}.$

Perform the indicated additions and subtractions in the following exercises:

**94.** $\frac{10}{4x+24}+\frac{3}{6x+36}.$

**95.** $\frac{y}{x(x+y)}+\frac{1}{x-y}.$

**96.** $\frac{a-b}{a^2+ay}-\frac{a}{a^2-y^2}.$

**97.** $\frac{2}{b}-\frac{2a}{a-b}-\frac{2ab}{(a-b)^2}.$

**98.** $\frac{2}{a-1}+\frac{2}{a+1}-\frac{4a}{a^2-2a+1}.$

**99.** $\frac{2a+3}{a+3}+\frac{a-4}{a-5}-\frac{3a^2-8a-27}{a^2-2a-15}.$

**100.** $1+\frac{1}{1+x}+\frac{1}{1-x}.$

**101.** $\frac{1}{1-x}-1+\frac{1}{1+x}.$

**102.** $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}.$

**103.** $\frac{1}{a+b}-\frac{1}{a-b}+\frac{1}{a-\frac{b}{a}}.$

Perform the multiplication indicated in each of the following exercises.

**104.** $\left(x-\frac{y}{z}\right)\left(x+\frac{y}{z}\right).$

**105.** $\left(ab+\frac{ab}{a-b}\right)\left(ab-\frac{ab}{a+b}\right).$

**106.** $\frac{a^2-b^2}{x+y}\cdot\frac{x^2-y^2}{a+b}\cdot\frac{c^2}{x-y}.$

**107.** $\frac{x^2-5x-24}{x^2-5x+6}\cdot\frac{x^2-2x-8}{x^2-4x-21}\cdot\frac{x^2-9x+14}{x^2-6x-16}.$

**108.** $\frac{r^4-s^4}{(r-s)^2}\cdot\frac{r-s}{r^2+rs}\cdot\frac{r^2}{r^2+s^2}.$

**109.** $\frac{x(x-y)}{x^2+2xy+y^2}\cdot\frac{x(x+y)}{x^2-2xy+y^2}.$

**110.** $\left(1-\frac{a-1}{a^2+6a+5}\right)\left(1-\frac{2}{a^2+7a+12}\right)$.

Perform the operations indicated in each of the following exercises.

**111.** $\frac{a^4-b^4}{a^2-2ab+b^2}\div\frac{a^2+b^2}{a^2-ab}$.

**112.** $\left(a+\frac{1}{b}\right)\div\left(a-\frac{1}{b}\right)$.

**113.** $(6a+3)\div\frac{4a+2}{3a}$.

**114.** $\left(\frac{2a+b}{a+b}+1\right)\div\left(\frac{2a+b}{a+b}-1\right)$.

**115.** $\frac{x^2-13x+22}{x^2-9x+8}\div\frac{x^2-5x+6}{x^2-6x-16}$.

**116.** $(r+s)\div\left(\frac{r^2-s^2}{1+y}\div\frac{r-s}{1-y^2}\right)$.

**117.** $\left(\frac{y^2}{1-y^2}+\frac{2y}{1-y}\right)\div\frac{1-y}{y+1}$.

**118.** $\left(\frac{2a(a^2-r^2)}{5b(d^2-r^2)}\div\frac{a^2-ar}{bd+br}\right)\cdot\frac{a^2+2ar+r^2}{d^2-2dr+r^2}$.

**119.** $\frac{\frac{a}{b^2}}{1-\frac{a^2}{b^2}}$.

**120.** $\frac{1+\frac{1}{x^2}+\frac{1}{x^4}}{1+\frac{1}{x}+\frac{1}{x^2}}$.

**121.** $\left[\frac{1-\frac{b}{a}+\frac{b^2}{a^2}}{1+\frac{b}{a}+\frac{b^2}{a^2}}\cdot\frac{\frac{a^3}{b^3}-1}{\frac{a^3}{b^3}+1}\right]\div\frac{\left(\frac{1}{a}-\frac{1}{b}\right)^2}{\left(\frac{1}{a}+\frac{1}{b}\right)^2}$.

## VI. RADICALS

Combine the following radicals.

**122.** $\sqrt{8}+\sqrt{18}+\sqrt{32}$. Check your answer by use of the table; at is, show that $\sqrt{8}+\sqrt{18}+\sqrt{32}$, as computed by the table, has the me value as your answer, similarly computed.

**123.** $\sqrt{108}+\sqrt{27}-\sqrt{75}$. Check your answer as in Ex. 122.

**124.** $\sqrt[3]{128}+\sqrt[3]{16}-\sqrt[3]{54}$. Check as in Exs. 122 and 123.

**125.** $\sqrt{72}+\sqrt{32}-\sqrt{50}$.

**126.** $\sqrt{\frac{3}{4}}+\sqrt{\frac{27}{16}}+\sqrt{\frac{75}{9}}$.

**127.** $\sqrt[3]{24}+\sqrt[3]{81}+\sqrt[3]{192}$.

**128.** $\sqrt{32a^2}-\sqrt{8a^2}+\sqrt{18a^2}$.

**129.** $\sqrt[3]{16a^3b^3}+\sqrt[3]{54a^3b^3}$.

**130.** $\sqrt{32a}-\sqrt{8a}+\sqrt{18a}$.

**131.** $\sqrt{2(x-y)^2}+\sqrt{8(x-y)^2}+\sqrt{18(x-y)^2}$.

**132.** $\sqrt{2(x-y)}+\sqrt{8(x-y)}+\sqrt{18(x-y)}$.

**133.** $\sqrt{\frac{1}{2}}+\sqrt{12\frac{1}{2}}+\sqrt{\frac{1}{8}}+\sqrt{1\frac{1}{8}}$.

**134.** $2\sqrt{3}-\frac{1}{2}\sqrt{12}+3\sqrt{27}$.

Find the following products, simplifying results as far as possible.

**135.** $\sqrt{3} \cdot \sqrt{27}$.

**136.** $\sqrt{8} \cdot \sqrt{12}$.

**137.** $\sqrt{6} \cdot \sqrt{4}$.

**138.** $\sqrt{7} \cdot \sqrt{9}$.

**139.** $\sqrt{10} \cdot \sqrt{3} \cdot \sqrt{2}$.

**140.** $\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{2}}$

**141.** $\sqrt{7} \cdot \sqrt{\frac{1}{7}}$.

**142.** $\sqrt{.1} \cdot \sqrt{.01}$.

**143.** $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$.

**144.** $(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})$.

**145.** $(\sqrt{6}-\sqrt{3})^2$.

**146.** $(\sqrt{7}-1)^2$.

**147.** $(3\sqrt{3}+2\sqrt{5})(\sqrt{3}-3\sqrt{5})$.

**148.** $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3})$.

**149.** $\sqrt{a} \cdot \sqrt{a^3}$.

**150.** $\sqrt{ab} \cdot \sqrt{a^3b^3}$.

**151.** $\sqrt{x^2y} \cdot \sqrt{xy}$.

**152.** $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.

**153.** $(\sqrt{a}-\sqrt{b})^2$.

**154.** $(\sqrt{2x}+\sqrt{9y})^2$.

Express each of the following quotients as a fraction under one radical sign, and reduce your answer to simplest form.

**155.** $\dfrac{\sqrt{15}}{\sqrt{3}}$.

SOLUTION. $\dfrac{\sqrt{15}}{\sqrt{3}}=\sqrt{\dfrac{15}{3}}=\sqrt{5}$. *Ans.*

**156.** $\dfrac{\sqrt{30}}{\sqrt{5}}$.

**157.** $\dfrac{\sqrt{3}}{\sqrt{12}}$.

**158.** $\dfrac{\sqrt{2}}{\sqrt{6}}$. **159.** $\dfrac{\sqrt{6}}{\sqrt{3}}$. **160.** $\dfrac{\sqrt{x^5}}{\sqrt{x^3}}$. **161.** $\dfrac{\sqrt{a^2-b^2}}{\sqrt{a+b}}$. **162.** $\dfrac{\sqrt{3(x^2-y^2)}}{\sqrt{12(x+y)^3}}$.

**163.** $\sqrt{x^2+x-20} \div \sqrt{x+5}$.

**164.** $\sqrt{81+9a^2+a^4} \div \sqrt{a^2-3a+9}$.

**165.** $\sqrt{x^6-y^6} \div \sqrt{x^3+y^3}$.

**166.** $\sqrt{x^7+2x^6-2x^4+2x^2-1} \div \sqrt{x+1}$.

**167.** $\sqrt{r^{2n}+11r^n+30} \div \sqrt{r^n+6}$.

# ANSWERS

**Page 1.** **1.** $x-y+z$. **2.** $x+y+z$. **3.** $2-a-b$. **4.** $m-n+2a$. **6.** $0$. **7.** $xy+4y^2-x^2$. **8.** $5m-2n$. **9.** $5a^2b+b^2c-4a^2c^2-8a^2b^2+3a^2c$. **10.** $\frac{x+2y}{-a+c}$. **11.** $4ab$. **12.** $\frac{-a^2-b^2}{2xy-y^2}$. **13.** $2ab-2ac$.

**Page 3.** **1.** $\frac{14}{21}$. **2.** $\frac{100}{125}$. **3.** $\frac{30a}{42}$. **4.** $\frac{16a^2y^2}{20y^3}$. **5.** $\frac{x^2-4x+3}{(x-1)^2}$. **6.** $\frac{3a+a^2}{9-a^2}$. **7.** $\frac{c-d}{a-b}$. **8.** $\frac{y}{a}$. **9.** $\frac{a^2x}{by^2}$. **10.** $\frac{2nx^2}{5amyz}$. **11.** $\frac{7a^4x^5y}{11b^2c^7}$. **12.** $\frac{x^m}{y^m}$. **13.** $\frac{a-b}{a+b}$. **14.** $\frac{a-b}{a+b}$. **15.** $\frac{3}{ab-b^2}$. **16.** $\frac{3a+3b}{2}$. **17.** $\frac{a^2c-b^2c}{3}$. **18.** $\frac{a(a+2b)^2}{b(a-2b)^2}$. **19.** $\frac{1-x^2}{1+x^2}$. **20.** $\frac{1-m}{1-n}$. **21.** $\frac{a-d}{m}$.

**Page 5.** **1.** $\frac{29x}{20}$. **2.** $\frac{25x-61}{56}$. **3.** $\frac{a-c}{ac}$. **4.** $\frac{a^2}{a-x}$. **5.** $\frac{2a^2}{a-b}$. **6.** $\frac{b-a}{ab}$. **7.** $\frac{4ab}{a^2-b^2}$. **8.** $\frac{2y^2}{y-x}$. **9.** $\frac{6x-4}{x^2-4}$. **10.** $\frac{x+2}{24}$. **11.** $\frac{x^2-ax+bx-ab}{x^2}$. **12.** $\frac{5a^2+26a-91}{(a+3)(a-3)(a+5)}$. **13.** $\frac{2n^2}{n-m}$. **14.** $\frac{6a-7}{a^2-4}$. **15.** $\frac{a^2+4ab+b^2}{a^2-b^2}$. **16.** $\frac{4a^2b}{a^4-b^4}$. **17.** $\frac{1+x^2}{x+x^2}$.

**Page 6.** **1.** $\frac{3x^2}{4cy}$. **2.** $\frac{-5bn}{4my}$. **3.** $\frac{5ab}{3xy}$. **4.** $\frac{3ax}{2b^n}$. **5.** $\frac{ab}{a^2-b^2}$. **6.** $\frac{5y}{4x}$. **7.** $\frac{b}{a}$. **8.** $\frac{2x-y}{2}$. **9.** $\frac{x^2+4x+4}{x^2+5x+4}$. **10.** $\frac{m+2}{m}$. **11.** $\frac{72a^2bc}{25x}$. **12.** $\frac{a}{bc^2}$. **13.** $\frac{7ax}{6y^5}$. **14.** $\frac{(m-y)^2}{y(m+y)}$. **15.** $10a$. **16.** $\frac{x+y}{x^2-4y^2}$. **17.** $\frac{x+5}{x-3}$. **18.** $\frac{1}{xy}$. **19.** $\frac{1}{b^2}$. **20.** $\frac{1}{1-x}$. **21.** $\frac{x^2y}{x+y}$. **22.** $\frac{x^2+2}{(x-1)(x+2)}$. **23.** $\frac{y^2-y+1}{y^2}$. **24.** $\frac{b}{x-y}$. **25.** $\frac{x^2-y^2}{x^2+y^2}$. **26.** $x-1$. **27.** $x+y$.

**Page 8.** **1.** 3. **2.** 5. **3.** 2. **4.** −7. **5.** 7. **6.** 1. **7.** 14 and 24. **8.** 52 and 44. **9.** $5 in the first bank, $40 in the second. **10.** 10 miles. **11.** 15 miles and 21 miles. **12.** 6 hours. **13.** 12 hours. **14.** 7½ hours. **15.** 120 miles per hour. **16.** 3 gallons. **17.** 16¼ pounds. **18.** 12¼ miles.

**Page 11.** **1.** $x=10$, $y=6$. **2.** $x=6$, $y=-2$. **3.** $x=3$, $y=8$. **4.** $x=-2$, $y=3$. **5.** $x=-11$, $y=4$. **6.** $x=9$, $y=6$. **7.** $x=12$, $y=14$. **8.** $x=16$, $y=12$. **9.** $x=18$, $y=6$. **10.** $x=3$, $y=2$. **11.** $x=4$, $y=5$. **12.** $x=\frac{2}{3}$, $y=\frac{15}{16}$. **13.** $x=\frac{1}{11}$, $y=\frac{2}{3}$. **14.** $x=5$, $y=1$. **15.** 40 and 35. **16.** 18 and 12. **17.** Father's 60, son's 40. **18.** \$1600 at 6%; \$900 at 5%. **19.** A, 18 days; B, 36 days. **20.** 26 \$1 bills, 12 \$2 bills. **21.** 16⅔ pounds of the 26-cent grade, 33⅓ pounds of the 35-cent grade. **22.** 15 gallons from first cask, 6 gallons from second. **23.** Automobile, 20 miles per hour, bicycle 14 miles per hour. **24.** Length 5 feet, breadth 3 feet.

**Page 19.** **1.** $x=\frac{b+1}{a}$. **2.** $x=\frac{c}{a+b}$. **3.** $x=-2b$. **4.** $x=-ab$. **5.** $x=\frac{ac}{c-1}$. **6.** $x=\frac{2b-3c-12}{1-b-c}$. **7.** $ab+7$. **8.** $a+3b$. **9.** $\frac{1}{a+b}$. **10.** $\frac{ac}{a-b}$. **11.** $x=\frac{6a+20b}{19}$, $y=\frac{4a-12b}{19}$. **12.** $x=\frac{2d+3b}{ad+bc}$, $y=\frac{3a-2c}{ad+bc}$. **13.** $x=\frac{dm-bn}{ad-bc}$, $y=\frac{an-cm}{ad-bc}$. **14.** $x=\frac{3b}{5}$, $y=-\frac{2a}{5}$. **15.** $x=-(a+b)$, $y=a+b$. **16.** *One part* $=\frac{am}{1+m}$, *other part* $=\frac{a}{1+m}$. **17.** $\frac{ab}{a+b}$. **18.** 20 feet. **19.** 3.74 inches. **20.** 11 minutes. **21.** $\frac{vt}{60T-t}$ miles per hour. **22.** 4 feet. **23.** 160 (approximately). **24.** (*a*) 1139.02 feet per second; (*b*) $58.3^{\circ+}$. **25.** (*a*) $N=\frac{R}{\pi d}$; (*b*) $N=10m+2\frac{1}{2}n+\frac{1}{10}q$; (*c*) $100c+10b+a$. **26.** (*a*) 2 pounds, 13 ounces; (*b*) 12 pounds, 13 ounces; (*c*) 8 ft. per sec.

**Page 23.** **1.** 256. **2.** $-1$. **3.** $\frac{64}{729}$. **4.** $x^{12}$. **5.** $q^{m+4}$. **6.** $z^{2r}$. **7.** 8. **8.** $\frac{16}{81}$. **9.** $x^8$. **10.** $q^{m-4}$. **11.** $z^2$. **12.** 64. **13.** 64. **14.** $x^{24}$. **15.** $a^6b^9$. **16.** $x^4y^4$. **17.** $m^6m^9w^3$. **18.** $(a+b)^8(c+d)^{12}$. **19.** $\frac{m^{20}}{n^4}$. **20.** $\frac{x^9}{y^9}$. **21.** $(-1)^k\frac{x^{2kn}}{y^{3km}}$. **22.** $\frac{1}{9}$. **23.** $\frac{1}{18}$. **24.** $\frac{1}{27}$. **25.** $-\frac{1}{2}$. **26.** $\frac{1}{a^7}$. **27.** $\frac{b}{n}$. **28.** $ab^{\frac{1}{2}}+a^{\frac{1}{2}}b$. **29.** $x^2$. **30.** $x^{-2}$. **31.** $\frac{1}{2}x^3$. **33.** 2. **34.** 3. **35.** $-2$. **36.** 9. **37.** 27. **38.** 2. **39.** $x^2$. **40.** $y^4$. **41.** $\sqrt[3]{4}$. **42.** $\sqrt[3]{m^2}\sqrt[4]{n^3}$. **44.** (*a*) 4, (*b*) 9, (*c*) $-32$, (*d*) $\frac{1}{(-3)^n}$, (*e*) $-8$, (*f*) $\frac{8}{125}$. **46.** $2a^{\frac{3}{2}}-a+9$.

**47.** $6x^2-7x^{\frac{5}{3}}-19x^{\frac{4}{3}}+5x+9x^{\frac{2}{3}}-2x^{\frac{1}{3}}$. **48.** $2-4a^{-\frac{4}{3}}x^{\frac{3}{2}}+2a^{-\frac{8}{3}}x^3$.

**49.** $5x^{\frac{2}{3}}-3x^{\frac{1}{3}}+1$. **50.** $x^{-\frac{1}{2}}-2x^{-\frac{1}{3}}+3x^{-\frac{1}{6}}-1$.

**51.** $x^{\frac{2}{3}}y^{-1}-3x^{\frac{1}{3}}y^{-\frac{1}{2}}+2-4x^{-\frac{1}{3}}y^{\frac{1}{2}}$.

**Page 25.** **1.** $3\sqrt{2}$. **2.** $2\sqrt{6}$. **3.** $6\sqrt{2}$. **4.** $5\sqrt{5}$. **5.** $3\sqrt{11}$.
**6.** $2\sqrt[3]{4}$. **7.** $3\sqrt[3]{2}$. **8.** $3\sqrt[3]{3}$. **9.** $2\sqrt[4]{2}$. **10.** $\frac{6\sqrt{2}}{5\sqrt{3}}$. **11.** $\frac{2\sqrt[3]{3}}{3\sqrt[3]{5}}$.
**12.** $6a^2b\sqrt{ab}$. **13.** $9m^2n^3\sqrt{mn}$. **14.** $2(a+b)\sqrt{a+b}$. **15.** $3xy\sqrt[3]{xz^2}$.
**16.** [illegible]. **17.** $\frac{2k\sqrt[3]{2h^2k}}{\text{[illegible]}\sqrt[3]{l}}$. **18.** $\frac{(a+b)c\sqrt{3d}}{2\sqrt{a^2-b^2}}$. **20.** $\frac{2\sqrt{7}+\sqrt{35}}{14}$.
**21.** $\frac{2\sqrt{3}-\sqrt{2}}{4}$. **22.** $\frac{11-6\sqrt{2}}{7}$. **23.** $\frac{6a+\sqrt{ab}-12b}{4a-9b}$. **24.** $\frac{x+\sqrt{x+1}-5}{x-3}$.
**25.** $\frac{6a+6+5\sqrt{2a^2+a}}{14a-9}$. **26.** $\frac{\sqrt{ab+b^2}-\sqrt{ab}}{b}$.

**Page 27.** **3.** 1. **4.** $4\sqrt{-2}$. **5.** $\frac{3+\sqrt{-2}}{11}$. **6.** $\frac{1-4\sqrt{-3}}{7}$.
**7.** $\frac{a^2-b^2+2ab\sqrt{-1}}{a^2+b^2}$. **10.** No.

**Page 33.** **1.** $-1\pm\sqrt{2}$. **2.** $2, -8$. **3.** $-2, 10$. **4.** $1, -\frac{7}{3}$. **5.** $2, -\frac{4}{5}$
**6.** $2, -\frac{13}{3}$. **7.** $-\frac{4}{3}, \frac{4}{5}$. **8.** $3, \frac{5}{3}$. **9.** $\frac{1}{6}(5\pm\sqrt{13})$. **10.** $\frac{1}{3}(3\pm\sqrt{5})$.
**11.** $\frac{2}{3}, -\frac{5}{6}$. **12.** $3, -\frac{6}{5}$. **13.** $3, -1$. **14.** $1, -\frac{10}{9}$. **15.** $2, -5$.
**16.** $\frac{1}{4}(-5\pm\sqrt{-7})$. **17.** $\frac{1}{6}(7\pm\sqrt{-11})$. **18.** $-1, -3$. **19.** $\pm\sqrt{-1}$.
**20.** $\pm 1$.

**Page 35.** **1.** $2a, -6a$. **2.** $3b, -7b$. **3.** $\frac{3a}{2}, -\frac{2a}{3}$. **4.** $3b, -7b$.
**5.** $\frac{5cd}{3}, -3cd$. **6.** $\frac{1}{a}, -\frac{6}{a}$. **7.** $m(-1\pm\sqrt{2})$ **8.** $-m\pm\sqrt{m^2+m}$. **9.** $a, 1$.
**10.** $a, -\frac{1}{a}$. **11.** $a, -\frac{a}{a+1}$. **12.** $1, -\frac{a+b}{b}$. **13.** $a+1, a-1$.
**14.** $\frac{b-2}{2}, \frac{b+2}{2}$. **15.** $a+b, \frac{1}{2}$. **16.** $-a, -b$. **17.** $\frac{1}{ab^2}, \frac{1}{a^2b}$
**18.** $\frac{a+b}{c}, \frac{c}{a+b}$. **19.** $a, -\frac{a}{7}$. **20.** $a+b, \frac{a+b}{ab}$.

**Page 37.** **1.** $-2, -3$. **2.** $9, -3$. **3.** $\frac{5}{3}, -\frac{3}{2}$. **4.** $\frac{3}{4}, -2$. **5.** $\frac{3}{4}, -\frac{1}{3}$
**6.** $\frac{2}{3a}, -\frac{4}{a}$. **7.** $\pm 2, \pm 2\sqrt{-1}$. **8.** $3, \frac{3}{2}(-1\pm\sqrt{-3})$. **9.** $\pm 1, \pm 2$
**10.** $\frac{1}{2}, \frac{1}{5}(-2\pm\sqrt{19})$. **11.** $-1, \frac{5}{3}$. **12.** $1, \frac{1}{4}(-2\pm 3\sqrt{-2})$
**13.** $-1, \pm\frac{1}{2}\sqrt{2}$. **14.** $a, b$.

**Page 39.** **1.** $\pm 1, \pm 2$. **2.** $\pm 2, \pm\sqrt{3}$. **3.** $\pm 1, \pm\frac{3}{2}$.
**4.** $1, \frac{2}{3}, \frac{1}{2}(-1\pm\sqrt{-3}), \frac{1}{3}(-1\pm\sqrt{-3})$. **5.** $3, -1$. **6.** $\pm 2, \pm\sqrt{-2}$.
**7.** $16, \frac{1}{81}$. **8.** 9. **9.** 12. **10.** $2, -\frac{2}{3}, \frac{1}{8}(3\pm\sqrt{57})$. **11.** $1, \frac{1}{4}(1\pm\sqrt{-15})$.
**12.** 25. **13.** 2. **14.** $27, -\frac{1}{8}$. **15.** $1, \frac{9}{49}$. **16.** $\frac{7}{2}, -\frac{3}{2}$. **17.** $\pm\frac{7}{6}$.

**Page 40.** **1.** $-1$. **2.** 8. **3.** $-\frac{8}{9}$. **4.** $4, -\frac{4}{7}$. **5.** $2a^2$. **6.** 0, 5
**7.** $2, \frac{2}{3}$. **8.** $a, b$.

**Page 40.** **1.** 8, 12. **2.** 5, 6. **3.** 20 rods by 8 rods. **4.** 2 in. **5.** 12. **6.** 0.41 in. **7.** 30 mi. per hr. **8.** 5 mi. per hr. **9.** 15. **10.** 30 min., 45 min. **11.** 20 in. **12.** 5.828 sq. ft. **13.** 5 in., 12 in. **14.** 108.3 yds., 51.7 yds. **15.** 2.89 hrs. after noon, 2.53 hrs. before noon. **17.** 7 sec.
**18.** 7 sec. **19.** About 238 ft. **20.** $L=\frac{1}{6}(3s\pm\sqrt{9s^2-96d^2})$.
**21.** $r=\frac{1}{2\pi}(-\pi h+\sqrt{\pi^2h^2+2\pi S})$.

**Page 43.** **2.** $a=2, b=-5, c=1$. **3.** $a=3, b=0, c=1$.
**4.** $a=2, b=2, c=-1$. **6.** $a=2, b=-(m+n), c=\frac{mn}{2}$.
**7.** $a=1, b=q-p, c=-pq$. **8.** $a=m^2+1, b=2bm, c=b^2-r^2$.
**9.** $a=4k^2-l^2, b=-(8k^2+2l^2), c=4k^2-l^2$.

**Page 45.** **1.** $-\frac{1}{2}$ and $-2$. **2.** $-\frac{2}{3}$ and $-3$. **3.** $\frac{2}{3}$ and $\frac{1}{2}$. **4.** $\frac{3}{2}$ and $-\frac{5}{2}$.
**5.** 5 and $-\frac{2}{3}$. **6.** $-\frac{3}{4}\pm\frac{1}{4}\sqrt{17}$. **7.** $-\frac{1}{3}\pm\frac{1}{3}\sqrt{13}$. **8.** $1\pm\frac{1}{3}\sqrt{3}$.
**9.** $3\pm\sqrt{-1}$. **10.** $m$ and $-n$. **11.** $-m$ and $n$. **12.** $2a$ and $3b$.
**13.** $-4m$ and $-3n$. **14.** $a(1\pm\sqrt{2})$. **15.** $-\frac{a}{2}$ and $\frac{b}{3}$.
**16.** $\frac{1}{2}(-p\pm\sqrt{p^2-4q})$.

**Page 48.** **1.** Real and unequal, rational. **2.** Real and unequal, rational. **3.** Imaginary. **4.** Real and unequal, irrational. **5.** Real and unequal, rational. **6.** Real and unequal, irrational. **7.** Real and unequal, irrational. **8.** Imaginary. **9.** Real and equal. **10.** Real and unequal, rational. **11.** Real and unequal, rational. **12.** Real and unequal, rational. **13.** Real and unequal, rational. **14.** Real and equal.
**16.**(*a*) $\frac{9}{2}$. **16.**(*b*) $-2$ and $-\frac{2}{3}$. **16.**(*c*) 1. **16.**(*d*) 4 and $-\frac{2}{3}$.
**17.** $\pm\sqrt{a^2m^2+b^2}$.

**Page 49.** **1.** $-2, -\frac{1}{3}$. **2.** $\frac{5}{2}, \frac{3}{2}$. **3.** 2, 1. **4.** $\frac{4}{5}, \frac{2}{5}$. **5.** $-\frac{7}{6}, -7$. **6.** $-\frac{1}{2}, \frac{1}{7}$. **7.** $-\frac{\sqrt{3}}{2}, -\frac{\sqrt{5}}{2}$. **8.** $-p, -q$. **10.**(*a*) 4, 3; **10.**(*b*) $-1, -1$. **10.**(*c*) 10, 13. **10.**(*d*) $\frac{1}{2}, \frac{1}{3}$. **10.**(*e*) $\sqrt{2}, \sqrt{5}$. **10.**(*f*) $\frac{5}{2}, \frac{3}{2}$. **10.**(*g*) $-\frac{1}{9}, -\frac{\sqrt{5}}{3}$.

**Page 50.** **1.** $x^2-3x+2=0$. **2.** $x^2+3x+2=0$. **3.** $3x^2-10x+3=0$. **4.** $6x^2+5x+1=0$. **5.** $x^2+(\sqrt{2}+\sqrt{3})x+\sqrt{6}=0$. **6.** $x^2-\sqrt{2}x-4=0$. **7.** $2x^2-(1+2\sqrt{5})x+\sqrt{5}=0$. **8.** $4x^2-2(\sqrt{5}-1)x-\sqrt{5}=0$. **9.** $x^2-mx-6m^2=0$. **10.** $x^2-2ax+a^2-b^2=0$. **11.** $x^2-4x+2=0$. **12.** $x^2-4x+1=0$. **13.** $4x^2+12x+3=0$. **14.** $4x^2+4x-1=0$.

**Page 62.** **1.** $(x=7, y=2)$ and $(x=-2, y=-7)$. **2.** $(x=2, y=4)$ and $\left(x=-\frac{1}{3}, y=\frac{5}{3}\right)$. **3.** $(x=3, y=1)$ and $(x=-2, y=-4)$. **4.** $(x=1\pm\frac{1}{2}\sqrt{46}, y=-\frac{1}{2}\pm\frac{1}{4}\sqrt{46})$. **5.** $(x=-3\pm\sqrt{34}, y=-3\pm\frac{1}{2}\sqrt{34})$. **6.** $(x=4, y=3)$ and $\left(x=\frac{68}{3}, y=-\frac{19}{3}\right)$. **7.** $(x=3, y=2)$ and $(x=96, y=-29)$. **8.** $(x=2, y=1)$ and $\left(x=-\frac{13}{3}, y=-\frac{14}{5}\right)$. **9.** $(x=3, y=2)$ and $\left(x=\frac{4}{23}, y=\frac{20}{23}\right)$. **10.** $\left(x=\frac{5}{27}, y=\frac{25}{27}\right)$ and $(x=5, y=-1)$.

**Page 65.** **1.** $(x=3, y=1)$; $(x=-3, y=1)$; $(x=3, y=-1)$; $(x=-3, y=-1)$. **2.** $(x=4, y=1)$; $(x=4, y=-1)$; $(x=-4, y=1)$; $(x=-4, y=-1)$. **3.** $(x=\pm3, y=\pm5; x=\pm5, y=\pm3)$. **4.** $(x=-2, y=5)$; $(x=2, y=-5)$. **5.** $(x=9, y=5)$; $(x=5, y=9)$; $(x=-5, y=-9)$; $(x=-9, y=-5)$. **6.** $(x=7, y=3)$; $(x=-3, y=-7)$; $(x=3, y=7)$; $(x=-7)$ $y=-3)$. **7.** $(x=3, y=2)$; $(x=-2, y=-3)$; $(x=2, y=3)$; $(x=-3, y=-2)$. **8.** $(x=7, y=4)$; $(x=-7, y=-4)$; $(x=\frac{11}{2}\sqrt{2}, y=\frac{3}{2}\sqrt{2})$; $(x=-\frac{11}{2}\sqrt{2}, y=-\frac{3}{2}\sqrt{2})$. **9.** $(s=4, t=\pm1)$; $(s=-\frac{15}{4}, t=\pm\frac{1}{4}\sqrt{-15})$. **10.** $(x=2, y=3)$; $\left(x=-\frac{25}{9}, y=-\frac{25}{6}\right)$; $(x=\frac{1}{18}+\frac{1}{18}\sqrt{-1799}, y=-\frac{1}{12}-\frac{1}{12}\sqrt{-1799})$; $(x=\frac{1}{18}-\frac{1}{18}\sqrt{-1799}, y=-\frac{1}{12}+\frac{1}{12}\sqrt{-1799})$.

**Page 68.** **1.** $(x=4,\ y=3)$; $(x=3,\ y=4)$. **2.** $(x=7,\ y=1)$; $(x=1,\ y=7)$; $(x=-1,\ y=-7)$; $(x=-7,\ y=-1)$. **3.** $(x=5,\ y=2)$; $\left(x=-4,\ y=-\frac{5}{2}\right)$. **4.** $\left(x=\frac{5}{4},\ y=2\right)$; $\left(x=1,\ y=3\right)$. **5.** $(x=2,\ y=3)$; $\left(x=54,\ y=\frac{1}{3}\right)$. **6.** $(x=3,\ y=1)$; $(x=-3,\ y=-1)$. **7.** $(x=5,\ y=\pm 4)$; $(x=-5,\ y=\pm 4)$. **8.** $(x=6,\ y=8)$; $(x=8,\ y=6)$; $(x=-6,\ y=-8)$; $(x=-8,\ y=-6)$. **9.** $(x=12,\ y=3)$; $\left((x=-7,\ y=-\frac{7}{4}\right)$; $(x=\frac{3}{2}+\frac{3}{2}\sqrt{29},\ y=\frac{1}{2}+\frac{1}{2}\sqrt{29})$; $(x=\frac{3}{2}-\frac{3}{2}\sqrt{29},\ y=\frac{1}{2}-\frac{1}{2}\sqrt{29})$. **10.** $(x=\sqrt{11},\ y=0)$; $(x=-\sqrt{11},\ y=0)$; $(x=1,\ y=2)$; $(x=-1,\ y=-2)$. **11.** $(x=1,\ y=-1)$; $\left(x=-\frac{1}{2},\ y=\frac{1}{2}\right)$; $(x=-\frac{1}{2}+\frac{\sqrt{3}}{2},\ y=-1+\sqrt{3})$; $(x=-\frac{1}{2}-\frac{\sqrt{3}}{2},\ y=-1-\sqrt{3})$. **12.** $(x=3,\ y=2)$; $(x=2,\ y=-3)$; $(x=16,\ y=-24)$; $\left(x=-\frac{12}{7},\ y=-\frac{8}{7}\right)$. **13.** $\left(x=\frac{6}{\sqrt{7}},\ y=\frac{4}{\sqrt{7}}\right)$; $\left(x=-\frac{6}{\sqrt{7}},\ y=-\frac{4}{\sqrt{7}}\right)$ **14.** $(x=6,\ y=2)$; $(x=-6,\ y=-2)$; $(x=8\sqrt{-1},\ y=6\sqrt{-1})$; $(x=-8\sqrt{-1},\ y=-6\sqrt{-1})$. **15.** $\left(x=2a,\ y=-b;\ x=a+3b,\ y=\frac{b-a}{2}\right)$.

**Page 68.** **1.** 4 and 8. **2.** 81 and 1. **3.** 12 in. and 16 in. **4.** 16 rods long, 10 rods wide. **5.** 2 ft. and 1 ft. **6.** 6 ft. and 1 ft. **7.** Altitude = 25.29 in., Base = 12.64 in. **8.** Length = 96.883 ft., Width = 24.772 ft. **9.** Either increase the length by 7.38 ft. and diminish the width by .38 ft., or diminish the length by 3.38 ft. and increase the width by 10.38 ft. **10.** 15 days for the one man and 10 days for the other. **11.** Circumference of fore wheel = 10 ft.; circumference of rear wheel = 12 ft. **12.** Principal = $125, rate = 6%. **13.** Time = 3 hours, rate = 10 miles per hour.

**Page 73.** **1.** 36. **2.** $-36$. **3.** $11x-11y$. **4.** 165. **5.** 208. **6.** $82\frac{1}{2}$. **7.** (*a*) 370.3 ft., (*b*) 2318.4 ft. **8.** $2.60; $68.90. **9.** 2500. **10.** 72. **11.** 336. **12.** 43 ft. **13.** 246 in. **14.** 10. **15.** 55. **16.** 267 in. **20.** (*a*) 20 in., (*b*) $31\frac{3}{7}$ in. **21.** $a=-13,\ l=27$.

**Page 78.** **1.** 512. **2.** 32. **3.** $\frac{1}{128}$. **4.** $a^{11}x^{11}$. **5.** $\frac{1}{16}\sqrt{2}$. **6.** 510. **7.** 3906. **8.** $-\frac{341}{1024}$. **9.** $\frac{1-a^{20}}{1-a^2}$. **10.** 765. **11.** $\frac{255}{16}$. **12.** 2046. **13.** 3279. **14.** (*a*) 128, (*b*) 1024. **15.** $\frac{100^4}{75}=1333333\frac{1}{3}$ bu. **16.** 1364. **17.** $\frac{127}{128}$. **18.** 4 sec. **19.** 256. **26.** 12.7279.

**Page 82.** **1.** 3. **2.** $\frac{3}{2}$. **3.** $\frac{3}{4}$. **4.** $4\frac{4}{9}$. **5.** $\frac{9}{104}$. **6.** $\frac{4}{7}$. **7.** $\frac{3}{2}(\sqrt{3}+1)$. **8.** $\frac{2}{3}(3-\sqrt{6})$. **9.** $\frac{8\sqrt{3}}{10\sqrt{3}-5}$. **10.** 36 in. **11.** $16\frac{4}{11}$ min. after 3 o'clock.

**Page 85.** **1.** $\frac{17}{111}$. **2.** $\frac{5}{37}$. **3.** $\frac{181}{333}$. **4.** $\frac{169}{495}$. **5.** $\frac{19}{110}$. **6.** $1\frac{7}{33}$. **7.** $3\frac{71}{330}$. **8.** $5\frac{32}{999}$. **9.** $6\frac{5}{37}$. **10.** $34\frac{371}{990}$.

**Page 92.** **2.** 300 ft. **3.** 7⁺ sq. yd. **4.** 8. **5.** 6750. **6.** $876.56. **7.** $1\frac{11}{12}$ ohms. **8.** 12 in. **9.** 11 mi. **10.** 302 (approximately). **13.** $2-\sqrt{2}$ ft., or approximately 0.586 ft. **15.** 2. **17.** $1\frac{1}{2}$ ft. **18.** 33⁺%. **20.** About 12%.

**Page 115.** **1.** 2.3821. **2.** 8.5786 − 10. **3.** 0.7456. **4.** 8.0957 − 10. **5.** 111.83⁺. **6.** 155.214⁺. **7.** 670.33. **8.** 9.854. **9.** 5137.8. **10.** 5.11. **11.** 3493077000 (approximately). **12.** 1236 (approximately). **13.** 0.000685.

**Page 117.** **1.** 0.3273. **2.** 1.4842. **3.** 4.3187. **4.** 8.8859 − 10. **5.** 15.967⁺. **6.** 6.50. **7.** 89.52⁺. **8.** 1.201⁺. **9.** 371. **10.** 0.56825. **11.** 68.8. **12.** 1.0114. **13.** .7734.

**Page 119.** **1.** 6.0205. **2.** 1.4826. **3.** 6.4910 − 10. **4.** 6.2560. **5.** 686.29. **6.** 288.1. **7.** 288.9. **8.** 0.0001641. **9.** 119.59. **10.** 1.437. **11.** 19.089.

**Page 120.** **1.** 0.2408. **2.** 0.1647. **3.** 9.5607 − 10. **4.** 0.3172. **5.** 17.716. **6.** 1.628. **7.** 1.629. **8.** 0.6254. **9.** 0.605. **10.** 14.312. **11.** 9.16.

**Page 121.** **1.** 13285. **2.** 6162.85. **3.** 2258. **4.** 603. **5.** 4.072. **6.** 15.61 ft. **7.** 3.88 sq. in. **8.** 4207.27 ft.

**Page 122.** **1.** $x=1.66^{+}$. **2.** $x=6.323^{+}$. **3.** $0.913^{+}$. **4.** $x=-0.682^{+}$. **5.** $x=0.494^{+}$. **6.** $x=2$ or $2.18^{+}$. **7.** $x=1.709^{+}$, $y=3.270^{+}$. **8.** $x=1.198^{+}$, $y=1.3868$.

**Page 126.** **1.** $537.10. **2.** $320.70. **3.** $1014. **4.** $439.50. **5.** 17 years. **6.** 14.2 years. **7.** 5%. **11.** 4.83 years. **12.** $5000.

**Page 128.** **2.** $2206. **3.** $362. **4.** $4965. **5.** $77,260. **7.** $370. **8.** $3,367. **9.** $669. **10.** $593. **11.** $955. **12.** $835. **13.** 6.384%.

**Page 135.** **1.** $x^3+3x^2y+3xy^2+y^3$. **2.** $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. **3.** $x^3-3x^2y+3xy^2-y^3$. **4.** $a^4-4a^3b+6a^2b^2-4ab^3+b^4$. **5.** $32+80r+80r^2+40r^3+10r^4+r^5$. **6.** $a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7$. **7.** $g^5-15g^4+90g^3-270g^2+405g-243$. **8.** $a^{10}+5a^8x+10a^6x^2+10a^4x^3+5a^2x^4+x^5$. **9.** $a^8-4a^6x^2+6a^4x^4-4a^2x^6+x^8$. **10.** $16a^4+32a^3+24a^2+8a+1$. **11.** $x^5-15x^4y+90x^3y^2-270x^2y^3+405xy^4-243y^5$. **12.** $1+6x^2+15x^4+20x^6+15x^8+6x^{10}+x^{12}$. **13.** $1-8x+28x^2-56x^3+70x^4-56x^5+28x^6-8x^7+x^8$. **14.** $x^5-\frac{5}{2}x^4+\frac{5}{2}x^3-\frac{5}{4}x^2+\frac{5}{16}x-\frac{1}{32}$. **15.** $81a^8-108a^6+54a^4-12a^2+1$. **16.** $a^{10}+10a^9x+45a^8x^2+120a^7x^3+210a^6x^4+252a^5x^5+210a^4x^6+120a^3x^7+45a^2x^8+10ax^9+x^{10}$. **17.** $\frac{1}{x^7}+\frac{7}{x^6y}+\frac{21}{x^5y^2}+\frac{35}{x^4y^3}+\frac{35}{x^3y^4}+\frac{21}{x^2y^5}+\frac{7}{xy^6}+\frac{1}{y^7}$. **18.** $\frac{a^5}{x^5}-5\frac{a^3}{x^3}+10\frac{a}{x}-10\frac{x}{a}+5\frac{x^3}{a^3}-\frac{x^5}{a^5}$. **19.** $a^2+3a\sqrt[3]{a}\sqrt[4]{b^3}+3b\sqrt[3]{a^2}\sqrt{b}+b^2\sqrt[4]{b}$. **20.** $2\sqrt{2}+\frac{6}{x^2}+3\frac{\sqrt{2}}{x^4}+\frac{1}{x^6}$.

**Page 136.** **1.** $70a^4x^4$. **2.** $-56x^3y^5$. **3.** $672x^6$. **4.** $-2002m^5n^9$. **5.** $-252a^{10}b^{10}$. **6.** $42504x^{19}$. **7.** $462x$. **8.** $12870a^8$. **9.** $495x^{12}$. **10.** $-960\sqrt{2}$.

**Page 139.** **1.** $a^{\frac{2}{3}}+\frac{2}{3}a^{-\frac{1}{3}}x-\frac{1}{9}a^{-\frac{4}{3}}x^2+\frac{4}{81}a^{-\frac{7}{3}}x^3-\cdots$

**2.** $a^{-2}-2a^{-3}x+3a^{-4}x^2-4a^{-5}x^3+\cdots$ **3.** $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\cdots$. **4.** $\frac{1}{\sqrt[4]{2}}+\frac{x}{8\sqrt[4]{2}}+\frac{5x^2}{128\sqrt[4]{2}}+\frac{15x^3}{1024\sqrt[4]{2}}+\cdots$. **5.** $(2a)^{\frac{3}{4}}+\frac{3}{4}(2a)^{-\frac{1}{4}}b-\frac{3}{32}(2a)^{-\frac{5}{4}}b^2+\frac{5}{128}(2a)^{-\frac{9}{4}}b^3-\cdots$. **6.** $a^{-\frac{9}{4}}+\frac{3}{4}a^{-\frac{21}{4}}x^2+\frac{21}{32}a^{-\frac{33}{4}}x^4+\frac{77}{128}a^{-\frac{45}{4}}x^6+\cdots$. **7.** $2^{\frac{1}{5}}+\frac{1}{5}\cdot 2^{-\frac{4}{5}}x-\frac{2}{25}\cdot 2^{-\frac{9}{5}}x^2+\frac{6}{125}\cdot 2^{-\frac{14}{5}}x^3-\cdots$. **8.** $a^{\frac{1}{5}}+\frac{1}{5}a^{-\frac{4}{5}}x-\frac{2}{25}a^{-\frac{9}{5}}x^2+\frac{6}{125}a^{-\frac{14}{5}}x^3-\cdots$. **9.** $\frac{7}{2^8}a^{-\frac{9}{2}}x^5$.

**10.** $\frac{5}{2^7}a^{-\frac{7}{2}}x^4$. **11.** $\frac{2618}{3^8}a^{-\frac{20}{3}}x^6$. **12.** $\frac{374}{3^9}x^7$. **13.** $45a^{-11}x^8$.

**14.** $-\frac{143}{2^{16}}x^{-\frac{15}{2}}y^9$. **15.** $\frac{22}{3^6}(2a)^{-\frac{14}{3}}b^5$. **16.** $4+.125-.00194+.00006-\cdots=4.12311^+$. **17.** $5+.2-.004+.00016-\cdots=5.19616^+$. **18.** $2+.08333-.00347+.000241-\cdots=2.08008^+$. **19.** $2-.0625-.00292-.0002136-.0000183\cdots=1.934338^+$. **20.** $2+.0375-.001406+.0000791-.000005\cdots=2.036168^+$.

**Page 142.** **12.** $5x-1$. **13.** $x^2-2x+1$.

**Page 151.** **7.** $(3, -2)$. **8.** $(1, -2)$. **9.** $\left(1, \frac{35}{6}\right), \left(-4, -\frac{40}{3}\right)$. **10.** $\frac{1}{12}x^2+\frac{1}{3}x$.

**Page 157.** **1.** $7\frac{1}{2}, 7\frac{1}{2}$. **2.** $\frac{h}{2}, \frac{h}{2}$. **3.** $\frac{1}{2}$. **4.** 16 by 8. **5.** $\frac{a}{6}$. **6.** 20 by 40. **7.** Depth $=\frac{14}{3}\sqrt{6}$, breadth $=\frac{14}{3}\sqrt{3}$.

**Page 161.** **1.** 21. **2.** $-11$. **3.** 1. **4.** 56. **5.** $ah^2+bh+c$.

**Page 164.** **1.** $x^2-2x-1$; $-3$. **2.** $x^2-6x+15$; $-31$. **3.** $3x^3-3x^2+3x-2$; 3. **4.** $x^3-x^2-x-15$; 0. **5.** $ax+(ah+b)$; $(ah^2+bh+c)$.

**Page 166.** **1.** $2, -3, -\frac{1}{2}$. **2.** $4, -1, -\frac{1}{2}$. **3.** $5, -2\pm\sqrt{3}$. **4.** $2, -1, -1, -2$. **5.** $3, 5, \frac{1}{6}(-3\pm\sqrt{-3})$. **6.** $2, 3, \frac{1}{2}(-1\pm\sqrt{5})$. **7.** $ax^2+(ar+b)x+(ar^2+br+c)=0$.

**Page 169.** **1.** $x^2-10x+9=0$. **2.**(*a*) $x^3-18x^2+9x-27=0$. **2.**(*b*) $x^4-12x^2-8x+32=0$. **2.**(*c*) $x^3-x^2+x-4=0$. **2.**(*d*) $2x^4-27x^2+405=0$. **3.**(*a*) $x^3-2x^2+3x-9=0$. **3.**(*b*) $x^4-5x^3+6x^2-8x-32=0$. **3.**(*c*) $x^3-3x^2+72=0$. **3.**(*d*) $x^4+9x^2-135=0$. **3.**(*e*) $x^3-4x^2+4=0$. **5.**(*a*) $x^3-3x^2+2x=0$. **5.**(*b*) $2x^3-7x^2+7x-2=0$. **5.**(*c*) $2x^4+16x^3+45x^2+56x+23=0$. **5.**(*d*) $2x^4-16x^3+45x^2-48x+7=0$. **5.** (*e*) $x^4-11x^3+15x^2+175x-482=0$. **5.**(*f*) $x^5+5x^4+10x^3+10x^2+8x+5=0$.

**Page 173.** **1.** $\frac{1}{3}, -\frac{1}{2}(1\pm\sqrt{5})$. **2.** $1, -2, \frac{3}{2}$. **3.** $+2, 3, -3, -\frac{1}{2}$. **4.** $2, 5, \frac{3}{2}, -4$. **5.** $\frac{1}{4}, \frac{3}{2}, -\frac{1}{3}$. **6.** $\frac{2}{3}, -\frac{1}{3}, -\frac{1}{2}$. **7.** $\frac{1}{3}, 1, \frac{5}{3}$. **8.** $-\frac{1}{2}, 3\pm\sqrt{2}$. **9.** $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{2}, -\frac{3}{4}$.

**Page 180.** **1.** 2.154. **2.** 4.134. **3.** 0.264. **4.** $-3.532$. **5.** 1.733. **6.** 3.23 and 3.73. **7.** 4.6635. **8.** 2.511. **9.** $x=1.618$, $y=0.618$. **10.** 1.25 inches. **11.** 1.199 inches. **12.** 4.1572 inches. **13.** 1.071 feet. **14.** 1.119 feet. **15.** 0.63 feet.

**Page 183.** **1.** 336. **2.** 625. **3.** 96. **4.** 11,880. **5.** 24. **6.** 362,880. **7.** 15,120. **8.** 1956. **9.** 2160. **10.** 5871. **11.** 256; 24; 4. **12.** 16. **13.** 16. **14.** 81.

**Page 186.** **1.** 60. **2.** 360. **3.** 3,628,800. **4.** 36. **5.** 72. **6.** 8,640. **7.** 14,400. **8.** 72. **9.** $(6!)^7$. **10.** 72. **11.** 360.

**Page 189.** **1.** 1,140. **2.** 35. **3.** 1,287. **4.** 45. **5.** 28. **6.** 1969. **7.** 4,410. **8.** 386. **9.** 364. **10.** 4,751,836,375. **11.** 5726.

**Page 192.** **1.** 380. **2.** 13,824. **3.** 720. **4.** 105. **6.** 163. **7.** 20; 84; 371. **8.** 34650. **9.** 420. **10.** 66. **11.** 369,600. **12.** 15,400. **13.** 19,958,400. **14.** 200. **15.** 255. **16.** 120. **17.** 181. **18.** 1130. **19.** 1728. **20.** 63. **21.** 315.

**Page 195.** **1.** (*a*) $\frac{6}{13}$, (*b*) $\frac{4}{13}$, (*c*) $\frac{3}{13}$. **2.** $\frac{1}{286}$. **4.** $\frac{3}{7}$. **5.** (*a*) $\frac{1}{91}$ (*b*) $\frac{8}{455}$, (*c*) $\frac{1}{273}$. **6.** $\frac{1}{24}$. **7.** (*a*) $\frac{11}{4165}$, (*b*) $\frac{2197}{20825}$. **8.** $\frac{9}{38}$. **9.** (*a*) $\frac{1}{8}$ (*b*) $\frac{3}{8}$, (*c*) $\frac{1}{2}$. **10.** $\frac{6}{5525}$. **11.** \$1.25. **12.** \$1.

**Page 201.** **1.** $\frac{5}{36}$. **2.** $\frac{8}{16575}$. **3.** $\frac{16}{5525}$. **4.** $\frac{1}{6}$. **5.** $\frac{91}{216}$. **6.** $\frac{1}{7}$.

**Page 202.** **1.** $\frac{45,957}{50,000}$; $\frac{7,237}{50,000}$.

**Page 204.** **1.** 2. **2.** 22. **3.** 51. **4.** $2ab$. **5.** $2a$. **6.** $6b$.

**Page 206.** **1.** 2, −2. **2.** 4, 2. **3.** $-2\frac{2}{45}, \frac{33}{45}$. **4.** 9, 6. **5.** −11, 4. **6.** $\frac{am+bc}{a^2+b^2}, \frac{bm-ac}{a^2+b^2}$. **7.** $\frac{ap+n}{ab+1}, \frac{p-bn}{ab+1}$. **8.** $a, b$. **9.** $\frac{3b}{5}, \frac{2a}{5}$.

**Page 208.** **1.** 18. **2.** −146. **3.** −51. **4.** 16. **5.** $11x-131$. **6.** $6b-3a-28$. **7.** $acz+bfx+cdy-cex-afy-bdz$. **8.** $x^2-y^2$.

**Page 211.** **1.** 1, 2, 3. **2.** 16, 10, −5. **3.** 39, 21, 12. **4.** 8, 9, 12. **5.** 1, 3, 5. **6.** $\frac{7}{2}, \frac{11}{2}, \frac{3}{2}$. **7.** $a+b, a-b, 2a$. **8.** $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

**Page 214.** **3.** 70.

**Page 219.** **1.** −100. **2.** 30. **3.** 0.

**Page 222.** **1.** 110. **2.** −68. **3.** 0.

**Page 228.** **1.** $x=1, y=2, z=3, w=1$. **2.** $x=1, y=2, z=3, w=4$. **4.** $x=3, y=1$. **5.** $x=\frac{3}{2}, y=4, z=\frac{3}{2}$. **6.** $k=2$ or $-6$. **8.** $5x^5-2x^4-9x^2+6x-1=0$. **12.** $k=-8$ or $-\frac{61}{2}$.

**Page 234.** **17.** $(1, 2)$. **18.** $(-1, 1)$. **19.** $(1, -1)$. **20.** $(2, -1)$ or $(-1, -3)$. **21.** $5-i$. **22.** $1+5i$. **23.** $-7+3i$. **24.** $2-3i$. **25.** $1-i$. **26.** $-7-4i$. **27.** $2+6i$. **28.** $3i$.

**Page 240.** **2.** $7-i$. **3.** $7+i$. **4.** $13i$. **5.** $8\frac{2}{3}-10\frac{2}{3}i$. **6.** $2i$. **7.** $-5+12i$. **8.** $6+2i$. **9.** $4$. **10.** $-10$. **11.** $16+28i$. **13.** $\frac{8}{5}+\frac{1}{5}i$. **14.** $1+i$. **15.** $\frac{12}{13}+\frac{5}{13}i$. **16.** $-\frac{6}{17}-\frac{1}{17}i$. **17.** $-\frac{1}{2}+\frac{1}{2}i$. **18.** $-\frac{3}{5}+\frac{4}{5}i$. **19.** $\frac{2}{5}+\frac{1}{5}i$. **20.** $-1$. **21.** $-\frac{1}{2}+\frac{1}{2}i$. **22.** $3-i$. **23.** $\frac{1}{25}(18+i)$. **24.** $\frac{1}{50}(1+57i)$.

**Page 243.** **2.** $-8i$. **3.** $-64$. **4.** $6+6\sqrt{3}i$. **5.** $24\sqrt{3}i$. **6.** $1$. **7.** $-72+72\sqrt{3}i$. **8.** $-128-128i$. **9.** $32i$.

**11.** $\sqrt[5]{8}(\cos A+i\sin A)$, where $A=63°$, $135°$, $207°$, $279°$, or $351°$.

**12.** $\sqrt{2}(\cos A+i\sin A)$, where $A=15°$, $135°$, or $255°$.

**13.** $\sqrt[5]{8}(\cos A+i\sin A)$, where $A=9°$, $81°$, $153°$, $225°$, or $297°$.

**14.** $\sqrt[4]{12}(\cos A+i\sin A)$, where $A=15°$ or $195°$.

**15.** $8(\cos A+i\sin A)$, where $A=60°$ or $240°$.

**16.** $2(\cos A+i\sin A)$, where $A=20°$, $140°$, or $260°$.

**17.** $2\sqrt[5]{2}(\cos A+i\sin A)$, where $A=20°$, $92°$, $164°$, $236°$, or $308°$.

**18.** $\cos A+i\sin A$, where $A=80°$, $200°$, or $320°$.

**19.** $\cos A+i\sin A$, where $A=30°$, $150°$, or $270°$.

**20.** $\cos A+i\sin A$, where $A=18°$, $90°$, $162°$, $234°$, or $306°$.

**21.** $4(\cos A+i\sin A)$, where $A=0$, or $180°$.

**22.** $\cos A+i\sin A$, where $A=0$, $120°$, or $240°$.

**23.** $\cos A+i\sin A$, where $A=0$, $90°$, $180°$, or $270°$.

**24.** $2(\cos A+i\sin A)$, where $A=22\frac{1}{2}°$, $112\frac{1}{2}°$, $202\frac{1}{2}°$, or $292\frac{1}{2}°$.

**25.** $7,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i.$ **26.** $-1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i.$

**27.** $2, 2i, -2, -2i$.

**28.** $2(\cos A+i\sin A)$, where $A=0$, $72°$, $144°$, $216°$, or $288°$.

**29.** $1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i.$

**30.** $\frac{\sqrt{2}}{2}(1+i),\ \frac{\sqrt{2}}{2}(-1+i),\ \frac{\sqrt{2}}{2}(-1-i),\ \frac{\sqrt{2}}{2}(1-i).$

**31.** $\cos A+i\sin A$, where $A=36°$, $108°$, $180°$, $252°$, or $324°$.

## I. ADDITION AND SUBTRACTION

**Page 245.** **1.** $11a-b+c$. **2.** $5x-11y$. **3.** $20x^3-3x^2-16$.
**4.** $8(m+n)+3(q-r)$. **5.** $13a+5b-17c-9d$.
**6.** $2(m+2n-q)x+(p-r)q-qz+3w$. **7.** $2x+y$.
**8.** $15x^3+19x^2-3x+4$. **9.** $5a-4b-14c$. **10.** $a+b+2c+d$.
**11.** $4x^2+8x+16$. **12.** $a-a^2$. **13.** $4x^2-xy-3y^2$. **14.** $-7s-4w$.
**15.** $-6a-14$. **16.** $9x-1$. **17.** $-2x$. **18.** $21z-2w$.

## II. MULTIPLICATION

**19.** $-2+x+x^2-6x^3+x^4+2x^5$. **20.** $3a^3-5a^2+a+1$.
**21.** $x^3+3x^2-x-6$. **22.** $-3a^3+13a^2+14a-20$.
**23.** $6n^4-19n^3+24n^2+56n-32$. **24.** $x^5+x^4-3x^3+16x^2-8x+8$.
**25.** $x^3+3x^2y+3xy^2+y^3$. **26.** $a^3+b^3$. **27.** $m^4+m^2n^2+n^4$.
**28.** $32r^4+32r^3s+24r^2s^2+8rs^3-6s^4$. **29.** $A^4-2A^2B^2+B^4$.
**30.** $x^4y^4+x^2y^2+1$.

## III. DIVISION

**31.** $a^4$. **32.** $(3g)^7$. **33.** $(3ab)^4$. **34.** $\frac{1}{2}(p^2q)^4$. **35.** $-4xyz$ **36.** $\frac{1}{5}a^2c^2$.
**37.** $\frac{2}{3}r^2$. **38.** $-\frac{3}{2}ab(a+b)$. **39.** $3m^2p+6n^2p$. **40.** $-2x-4y+8z$.
**41.** $3x+1$. **42.** $5x+2$. **43.** $2y^2+2y+1$. **44.** $3x^2-2x-1$. **45.** $x+2$.
**46.** $2x-1+\dfrac{3}{x+2}$. **47.** $x^2-2x+1-\dfrac{x-1}{x^2-x-2}$. **48.** $a^2-ab+b^2$.
**49.** $x^3-x^2y+xy^2-y^3$.

## IV. FACTORING

**50.** (*a*) $x(x+2)$.
(*b*) $xy(x+y)$.
(*c*) $8a(a+3)$.
(*d*) $b^2(a+b-c)$.
(*e*) $5c^2dx^3(5+7cdx-11dx^2)$.
(*f*) $(3x-1)(m-n)$.
(*g*) $(a+b-c)(m+n-q)$.
(*h*) $(q-x)(p-r)$.
(*i*) $(y-4)(y+x)$.
(*j*) $(x^2-5)(3x-2y)$.

**51.** (*a*) $(9-x)(9+x)$.
(*b*) $(a-bc)(a+bc)$.
(*c*) $(12x-2)(12x+4)$.
(*d*) $(\frac{1}{2}xy-6)(\frac{1}{2}xy+6)$.
(*e*) $(3b-a+x)(3b+a-x)$.
(*f*) $(2a+4b)(12a-4b)$.
(*g*) $(7x+2)(-3x+8)$.
(*h*) $(x^2-x-2)(x^2+3x+2)$.

**52.** (*a*) $(x+4)(x+2)$.
(*b*) $(x-4)(x-2)$.
(*c*) $(y+7)(y-6)$.
(*d*) $(x-16)(x+3)$.
(*e*) $(x+8y)(x-7y)$.
(*f*) $(4+a)(3+a)$.
(*g*) $(y-7n)(y+4n)$.
(*h*) $(ax+4)(ax-3)$.

**53.** (*a*) $(x-4)^2$.
(*b*) $(x-6)^2$.
(*c*) Not a square.
(*d*) $(9x+1)^2$.
(*e*) $(9-4r)^2$.
(*f*) $(4y-3)^2$.
(*g*) $(2xy-5)^2$.
(*h*) $(2x-y)^2$.
(*i*) $(4-3a+3b)^2$.
(*j*) $(a-c)^2$.

54. $(x-1)(x-1)(x+1)$. 55. $2x(x-2y)^2$. 56. $(x+3a)(x-1)$.
57. $(a+2)(a+2)(a-2)$. 58. $(x-3)(x+3)(x-2)(x+2)$.
59. [illegible] $(x+y)$. 60. $(x-1)(x-1)(x^2+2x-1)$.
61. $(1-a+b)(1+a-b)$. 62. $(m-2n+4)(m-2n-4)$.
63. $(1-x-y)(1+x+y)$. 64. $(1+3c)^2$. 65. $(x-1)^2(x+2)^2$.
66. $(x-1)(x+1)(3+4x)$. 67. $(1-ab-xy)(1+ab+xy)$.
68. [illegible] 69. $(x+y)(x^2-xy+y^2+x-y)$.
70. $(x-1)($[illegible]$+2x^2+2x+1)$. 71. $(1-x)^2(1-2x-x^2)$.
72. $(3x-1)(2x+3)$. 73. $2(5x+1)(2x-1)$. 74. $(2x-5y)(4x+y)$.
75. $(x^2-2)(x^2+2)(x^2+1)$. 76. $(11x+1)(x+3)$. 77. $(5s-4t)(5s-t)$.
78. $(3x^2+5a)(3x^2-5a)$. 79. $(a-b+c)(a-b+c)$.
80. $(ax^2-b+y^2)(ax^2+b-y^2)$. 81. $(x-y)(x^2-xy+y^2+1)$.
82. $(x-3y+1)(x+3y+1)$. 83. $(a-2b)(a^2+4b^2)$. 84. $(2x^2-y)(8x^3+y)$.
85. [illegible]

## V. FRACTIONS

86. $\frac{a+3}{a-7}$. 87. $\frac{2x+3}{2x-3}$. 88. $\frac{x+1}{x-1}$. 89. $\frac{r-b}{c-k}$. 90. $\frac{a(a+2b)^2}{b(a-2b)^2(a-b)^2}$.
91. $2a-b$. 92. $\frac{dx-a+c+x}{d}$. 93. $\frac{1+2m+2m^2}{1+m}$. 94. $\frac{3}{x+6}$.
95. [illegible] 96. $\frac{ab+ay-by}{a(a^2-y^2)}$. 97. $\frac{2a^2-4ab+2b^2-2a^2b}{b(a-b)^2}$.
98. [illegible] 99. 0. 100. $\frac{3-x^2}{1-x^2}$. 101. $\frac{1+x^2}{1-x^2}$. 102. $\frac{2ab^2}{(a^2-b^2)(a^2+b^2)}$.
103. [illegible] 104. $x^2-\frac{y^2}{z^2}$. 105. $a^2b^2-\frac{a^2b^2}{(a-b)^2}$. 106. $c^2(a-b)$. 107. $\frac{x-4}{x-3}$.
108. $x$. 109. $\frac{x^2}{x^2-y^2}$. 110. $\frac{(a+2)^2}{(a+1)(a+4)}$. 111. $a(a+b)$. 112. $\frac{ab+1}{ab-1}$.
113. [illegible] 114. $\frac{3a+2b}{a}$. 115. $\frac{(x+2)(x-11)}{(x-3)(x-1)}$. 116. $\frac{1}{1-y}$. 117. $\frac{3y^2+2y}{(1-y)^2}$.
118. [illegible] 119. $\frac{a}{b^2-a^2}$. 120. $\frac{x^2-x+1}{x^2}$. 121. $\frac{a+b}{a-b}$.

## VI. RADICALS

122. $9\sqrt{2}$. 123. $4\sqrt{3}$. 124. $3\sqrt[3]{2}$. 125. $5\sqrt{2}$. 126. $\frac{35}{12}\sqrt{3}$.
127. $9\sqrt[3]{3}$. 128. $5a\sqrt{2}$. 129. $5ab\sqrt[3]{2}$. 130. $5\sqrt{2a}$. 131. $6(x-y)\sqrt{2}$.
132. $6\sqrt{2(x-y)}$. 133. $4\sqrt{2}$. 134. $10\sqrt{3}$. 135. 9. 136. $4\sqrt{6}$.
137. $2\sqrt{6}$. 138. $3\sqrt{7}$. 139. $2\sqrt{15}$. 140. $\frac{3}{4}\sqrt{2}$. 141. 1. 142. $.01\sqrt{10}$.

**143.** 1. **144.** 10. **145.** $9-6\sqrt{2}$. **146.** $8-2\sqrt{7}$. **147.** $-7\sqrt{15}-21$.
**148.** $-1+\sqrt{10}-\sqrt{15}$. **149.** $a^2$. **150.** $a^2b^2$. **151.** $xy\sqrt{x}$. **152.** $x-y$.
**153.** $a-2\sqrt{ab}+b$. **154.** $2x+6\sqrt{2xy}+9y$. **155.** $\sqrt{5}$. **156.** $\sqrt{6}$.
**157.** $\frac{1}{2}$. **158.** $\sqrt{\frac{1}{3}}$. **159.** $\sqrt{2}$. **160.** $x$. **161.** $\sqrt{a-b}$.
**162.** $\frac{\sqrt{x-y}}{2(x+y)}$. **163.** $\sqrt{x-4}$. **164.** $\sqrt{9+3a+a^2}$. **165.** $\sqrt{x^3-y^3}$.
**166.** $\sqrt{x^6+x^5-x^4-x^3+x^2+x-1}$. **167.** $\sqrt{r^n+5}$.

# APPENDIX

## TABLE OF POWERS AND ROOTS

### Explanation

**1. Square Roots.** The way to find square roots from the Table is best understood from an example. Thus, suppose we wish to find $\sqrt{1.48}$. To do this we first locate 1.48 in the column headed by the letter $n$. We find it near the bottom of this column (next to the last number). Now we go across on that level until we get into the column headed by $\sqrt{n}$. We find at that place the number 1.21655. This is our answer. That is, $\sqrt{1.48}=1.21655$ (approximately).

If we had wanted $\sqrt{14.8}$ instead of $\sqrt{1.48}$ the work would have been the same except that we would have gone over into the column headed $\sqrt{10\,n}$ (because $14.8=10\times 1.48$). The number thus located is seen to be 3.84708, which is, therefore, the desired value of $\sqrt{14.8}$.

Again, if we had wished to find $\sqrt{148}$ the work would take us back again to the column headed $\sqrt{n}$, but now instead of the answer being 1.21655 it would be 12.1655. In other words, the order of the digits in $\sqrt{148}$ is the same as for $\sqrt{1.48}$, but the decimal point in the answer is one place farther to the right.

Similarly, if we desired $\sqrt{1480}$ the work would be the same as before except that we must now use the column headed $\sqrt{10\,n}$ and move the decimal point there occurring one place farther to the right. This is seen to give 38.4708.

Thus we see how to get the square root of 1.48 or any power of 10 times that number.

In the same way, if we wish to find $\sqrt{.148}$, or $\sqrt{.0148}$, or $\sqrt{.00148}$, or the square root of any number obtained by dividing 1.48 by any power of 10, we can get the answers from the column headed $\sqrt{n}$ or $\sqrt{10\,n}$ by merely placing the decimal point properly. Thus, we find that $\sqrt{.148} = .384708$, $\sqrt{.0148} = .121655$, $\sqrt{.00148} = .0384708$, etc.

What we have seen in regard to the square root of 1.48 or of that number multiplied or divided by any power of 10 holds true in a similar way for *any* number that occurs in the column headed $n$, so that the tables thus give us the square roots of a great many numbers.

**2. Cube Roots.** Cube roots are located in the tables in much the same way as that just described for square roots, but we have here three columns to select from instead of two, namely the columns headed $\sqrt[3]{n}$, $\sqrt[3]{10\,n}$, $\sqrt[3]{100\,n}$.

*Illustration.*

$\sqrt[3]{1.48}$ occurs in the column headed $\sqrt[3]{n}$ and is seen to be 1.13960.

$\sqrt[3]{14.8}$ occurs in the column headed $\sqrt[3]{10\,n}$ and is seen to be 2.4552.

$\sqrt[3]{148}$ occurs in the column headed $\sqrt[3]{100\,n}$ and is seen to be 5.28957.

To get $\sqrt[3]{.148}$ we observe that $.148 = \sqrt[3]{\frac{1.48}{10}} = \sqrt[3]{\frac{148}{1000}} = \frac{1}{10}\sqrt[3]{148}$. Thus, we look up $\sqrt[3]{148}$ and divide it by 10. The result is instantly seen to be .528957. Similarly, to get $\sqrt[3]{.0148}$ we observe that $\sqrt[3]{.0148} = \sqrt[3]{\frac{1.48}{100}} = \sqrt[3]{\frac{14.8}{1000}} = \frac{1}{10}\sqrt[3]{14.8}$. Thus, we look up $\sqrt[3]{14.8}$ and divide it by 10, giving the result .24552.

To get $\sqrt[3]{.00148}$ we observe that $\sqrt[3]{.00148} = \sqrt[3]{\frac{1.48}{1000}} = \frac{1}{10}\sqrt[3]{1.48}$, so that we must divide $\sqrt[3]{1.48}$ by 10. This gives .11396.

Similarly the cube root of any number occurring in the column headed $n$ may be found, as well as the cube root of any number obtained by multiplying or dividing such a number by any power of 10.

**3. Squares and Cubes.** To find the square of 1.48 we naturally look at the proper level in the column headed $n^2$. Here we find 2.1904, which is the answer. If we wished the square of 14.8 the result would be the same except that the decimal point must be moved *two* places to the *right*, giving 219.04 as the answer. Similarly the value of $(148)^2$ is 21904.0 etc.

On the other hand, the value of $(.148)^2$ is found by moving the decimal point two places to the *left*, thus giving .021904. Similarly, $(.0148)^2 = .00021904$, etc.

To find $(1.48)^3$ we look at the proper level in the column headed $n^3$ where we find 3.24179. The value of $(14.8)^3$ is the same except that we must move the decimal point *three* places to the *right*, giving 3241.79. Similarly, in finding $(.148)^3$ we must move the decimal point three places to the *left*, giving .00324179.

### EXERCISES

Read off from the tables the values of each of the following expressions.

**1.** $\sqrt{41}$
**2.** $\sqrt{8.9}$
**3.** $\sqrt[3]{67}$
**4.** $\sqrt[3]{670}$
**5.** $\sqrt{.89}$
**6.** $\sqrt{.016}$
**7.** $\sqrt{93.7}$
**8.** $\sqrt[3]{93.7}$
**9.** $\sqrt{.00154}$
**10.** $\sqrt[3]{.00154}$
**11.** $\sqrt{.000143}$
**12.** $\sqrt[3]{.000143}$

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **1.00** | 1.0000 | 1.00000 | 3.16228 | 1.00000 | 1.00000 | 2.15443 | 4.64159 |
| 1.01 | 1.0201 | 1.00499 | 3.17805 | 1.03030 | 1.00332 | 2.16159 | 4.65701 |
| 1.02 | 1.0404 | 1.00995 | 3.19374 | 1.06121 | 1.00662 | 2.16870 | 4.67233 |
| 1.03 | 1.0609 | 1.01489 | 3.20936 | 1.09273 | 1.00990 | 2.17577 | 4.68755 |
| 1.04 | 1.0816 | 1.01980 | 3.22490 | 1.12486 | 1.01316 | 2.18279 | 4.70267 |
| 1.05 | 1.1025 | 1.02470 | 3.24037 | 1.15762 | 1.01640 | 2.18976 | 4.71769 |
| 1.06 | 1.1236 | 1.02956 | 3.25576 | 1.19102 | 1.01961 | 2.19669 | 4.73262 |
| 1.07 | 1.1449 | 1.03441 | 3.27109 | 1.22504 | 1.02281 | 2.20358 | 4.74746 |
| 1.08 | 1.1664 | 1.03923 | 3.28634 | 1.25971 | 1.02599 | 2.21042 | 4.76220 |
| 1.09 | 1.1881 | 1.04403 | 3.30151 | 1.29503 | 1.02914 | 2.21722 | 4.77686 |
| **1.10** | 1.2100 | 1.04881 | 3.31662 | 1.33100 | 1.03228 | 2.22398 | 4.79142 |
| 1.11 | 1.2321 | 1.05357 | 3.33167 | 1.36763 | 1.03540 | 2.23070 | 4.80590 |
| 1.12 | 1.2544 | 1.05830 | 3.34664 | 1.40493 | 1.03850 | 2.23738 | 4.82028 |
| 1.13 | 1.2769 | 1.06301 | 3.36155 | 1.44290 | 1.04158 | 2.24402 | 4.83459 |
| 1.14 | 1.2996 | 1.06771 | 3.37639 | 1.48154 | 1.04464 | 2.25062 | 4.84881 |
| 1.15 | 1.3225 | 1.07238 | 3.39116 | 1.52088 | 1.04769 | 2.25718 | 4.86294 |
| 1.16 | 1.3456 | 1.07703 | 3.40588 | 1.56090 | 1.05072 | 2.26370 | 4.87700 |
| 1.17 | 1.3689 | 1.08167 | 3.42053 | 1.60161 | 1.05373 | 2.27019 | 4.89097 |
| 1.18 | 1.3924 | 1.08628 | 3.43511 | 1.64303 | 1.05672 | 2.27664 | 4.90487 |
| 1.19 | 1.4161 | 1.09087 | 3.44964 | 1.68516 | 1.05970 | 2.28305 | 4.91868 |
| **1.20** | 1.4400 | 1.09545 | 3.46410 | 1.72800 | 1.06266 | 2.28943 | 4.93242 |
| 1.21 | 1.4641 | 1.10000 | 3.47851 | 1.77156 | 1.06560 | 2.29577 | 4.94609 |
| 1.22 | 1.4884 | 1.10454 | 3.49285 | 1.81585 | 1.06853 | 2.30208 | 4.95968 |
| 1.23 | 1.5129 | 1.10905 | 3.50714 | 1.86087 | 1.07144 | 2.30835 | 4.97319 |
| 1.24 | 1.5376 | 1.11355 | 3.52136 | 1.90662 | 1.07434 | 2.31459 | 4.98663 |
| 1.25 | 1.5625 | 1.11803 | 3.53553 | 1.95312 | 1.07722 | 2.32079 | 5.00000 |
| 1.26 | 1.5876 | 1.12250 | 3.54965 | 2.00038 | 1.08008 | 2.32697 | 5.01330 |
| 1.27 | 1.6129 | 1.12694 | 3.56371 | 2.04838 | 1.08293 | 2.33311 | 5.02653 |
| 1.28 | 1.6384 | 1.13137 | 3.57771 | 2.09715 | 1.08577 | 2.33921 | 5.03968 |
| 1.29 | 1.6641 | 1.13578 | 3.59166 | 2.14669 | 1.08859 | 2.34529 | 5.05277 |
| **1.30** | 1.6900 | 1.14018 | 3.60555 | 2.19700 | 1.09139 | 2.35133 | 5.06580 |
| 1.31 | 1.7161 | 1.14455 | 3.61939 | 2.24809 | 1.09418 | 2.35735 | 5.07875 |
| 1.32 | 1.7424 | 1.14891 | 3.63318 | 2.29997 | 1.09696 | 2.36333 | 5.09164 |
| 1.33 | 1.7689 | 1.15326 | 3.64692 | 2.35264 | 1.09972 | 2.36928 | 5.10447 |
| 1.34 | 1.7956 | 1.15758 | 3.66060 | 2.40610 | 1.10247 | 2.37521 | 5.11723 |
| 1.35 | 1.8225 | 1.16190 | 3.67423 | 2.46038 | 1.10521 | 2.38110 | 5.12993 |
| 1.36 | 1.8496 | 1.16619 | 3.68782 | 2.51546 | 1.10793 | 2.38697 | 5.14256 |
| 1.37 | 1.8769 | 1.17047 | 3.70135 | 2.57135 | 1.11064 | 2.39280 | 5.15514 |
| 1.38 | 1.9044 | 1.17473 | 3.71484 | 2.62807 | 1.11334 | 2.39861 | 5.16765 |
| 1.39 | 1.9321 | 1.17898 | 3.72827 | 2.68562 | 1.11602 | 2.40439 | 5.18010 |
| **1.40** | 1.9600 | 1.18322 | 3.74166 | 2.74400 | 1.11869 | 2.41014 | 5.19249 |
| 1.41 | 1.9881 | 1.18743 | 3.75500 | 2.80322 | 1.12135 | 2.41587 | 5.20483 |
| 1.42 | 2.0164 | 1.19164 | 3.76829 | 2.86329 | 1.12399 | 2.42156 | 5.21710 |
| 1.43 | 2.0449 | 1.19583 | 3.78153 | 2.92421 | 1.12662 | 2.42724 | 5.22932 |
| 1.44 | 2.0736 | 1.20000 | 3.79473 | 2.98598 | 1.12924 | 2.43288 | 5.24148 |
| 1.45 | 2.1025 | 1.20416 | 3.80789 | 3.04862 | 1.13185 | 2.43850 | 5.25359 |
| 1.46 | 2.1316 | 1.20830 | 3.82099 | 3.11214 | 1.13445 | 2.44409 | 5.26564 |
| 1.47 | 2.1609 | 1.21244 | 3.83406 | 3.17652 | 1.13703 | 2.44966 | 5.27763 |
| 1.48 | 2.1904 | 1.21655 | 3.84708 | 3.24179 | 1.13960 | 2.45520 | 5.28957 |
| 1.49 | 2.2201 | 1.22066 | 3.86005 | 3.30795 | 1.14216 | 2.46072 | 5.30146 |

# Powers and Roots

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **1.50** | 2.2500 | 1.22474 | 3.87298 | 3.37500 | 1.14471 | 2.46621 | 5.31329 |
| 1.51 | 2.2801 | 1.22882 | 3.88587 | 3.44295 | 1.14725 | 2.47168 | 5.32507 |
| 1.52 | 2.3104 | 1.23288 | 3.89872 | 3.51181 | 1.14978 | 2.47712 | 5.33680 |
| 1.53 | 2.3409 | 1.23693 | 3.91152 | 3.58158 | 1.15230 | 2.48255 | 5.34848 |
| 1.54 | 2.3716 | 1.24097 | 3.92428 | 3.65226 | 1.15480 | 2.48794 | 5.36011 |
| 1.55 | 2.4025 | 1.24499 | 3.93700 | 3.72388 | 1.15729 | 2.49332 | 5.37169 |
| 1.56 | 2.4336 | 1.24900 | 3.94968 | 3.79642 | 1.15978 | 2.49867 | 5.38321 |
| 1.57 | 2.4649 | 1.25300 | 3.96232 | 3.86989 | 1.16225 | 2.50399 | 5.39469 |
| 1.58 | 2.4964 | 1.25698 | 3.97492 | 3.94431 | 1.16471 | 2.50930 | 5.40612 |
| 1.59 | 2.5281 | 1.26095 | 3.98748 | 4.01968 | 1.16717 | 2.51458 | 5.41750 |
| **1.60** | 2.5600 | 1.26491 | 4.00000 | 4.09600 | 1.16961 | 2.51984 | 5.42884 |
| 1.61 | 2.5921 | 1.26886 | 4.01248 | 4.17328 | 1.17204 | 2.52508 | 5.44012 |
| 1.62 | 2.6244 | 1.27279 | 4.02492 | 4.25153 | 1.17446 | 2.53030 | 5.45136 |
| 1.63 | 2.6569 | 1.27671 | 4.03733 | 4.33075 | 1.17687 | 2.53549 | 5.46256 |
| 1.64 | 2.6896 | 1.28062 | 4.04969 | 4.41094 | 1.17927 | 2.54067 | 5.47370 |
| 1.65 | 2.7225 | 1.28452 | 4.06202 | 4.49212 | 1.18167 | 2.54582 | 5.48481 |
| 1.66 | 2.7556 | 1.28841 | 4.07431 | 4.57430 | 1.18405 | 2.55095 | 5.49586 |
| 1.67 | 2.7889 | 1.29228 | 4.08656 | 4.65746 | 1.18642 | 2.55607 | 5.50688 |
| 1.68 | 2.8224 | 1.29615 | 4.09878 | 4.74163 | 1.18878 | 2.56116 | 5.51785 |
| 1.69 | 2.8561 | 1.30000 | 4.11096 | 4.82681 | 1.19114 | 2.56623 | 5.52877 |
| **1.70** | 2.8900 | 1.30384 | 4.12311 | 4.91300 | 1.19348 | 2.57128 | 5.53966 |
| 1.71 | 2.9241 | 1.30767 | 4.13521 | 5.00021 | 1.19582 | 2.57631 | 5.55050 |
| 1.72 | 2.9584 | 1.31149 | 4.14729 | 5.08845 | 1.19815 | 2.58133 | 5.56130 |
| 1.73 | 2.9929 | 1.31529 | 4.15933 | 5.17772 | 1.20046 | 2.58632 | 5.57205 |
| 1.74 | 3.0276 | 1.31909 | 4.17133 | 5.26802 | 1.20277 | 2.59129 | 5.58277 |
| 1.75 | 3.0625 | 1.32288 | 4.18330 | 5.35938 | 1.20507 | 2.59625 | 5.59344 |
| 1.76 | 3.0976 | 1.32665 | 4.19524 | 5.45178 | 1.20736 | 2.60118 | 5.60408 |
| 1.77 | 3.1329 | 1.33041 | 4.20714 | 5.54523 | 1.20964 | 2.60610 | 5.61467 |
| 1.78 | 3.1684 | 1.33417 | 4.21900 | 5.63975 | 1.21192 | 2.61100 | 5.62523 |
| 1.79 | 3.2041 | 1.33791 | 4.23084 | 5.73534 | 1.21418 | 2.61588 | 5.63574 |
| **1.80** | 3.2400 | 1.34164 | 4.24264 | 5.83200 | 1.21644 | 2.62074 | 5.64622 |
| 1.81 | 3.2761 | 1.34536 | 4.25441 | 5.92974 | 1.21869 | 2.62559 | 5.65665 |
| 1.82 | 3.3124 | 1.34907 | 4.26615 | 6.02857 | 1.22093 | 2.63041 | 5.66705 |
| 1.83 | 3.3489 | 1.35277 | 4.27785 | 6.12849 | 1.22316 | 2.63522 | 5.67741 |
| 1.84 | 3.3856 | 1.35647 | 4.28952 | 6.22950 | 1.22539 | 2.64001 | 5.68773 |
| 1.85 | 3.4225 | 1.36015 | 4.30116 | 6.33162 | 1.22760 | 2.64479 | 5.69802 |
| 1.86 | 3.4596 | 1.36382 | 4.31277 | 6.43486 | 1.22981 | 2.64954 | 5.70827 |
| 1.87 | 3.4969 | 1.36748 | 4.32435 | 6.53920 | 1.23201 | 2.65428 | 5.71848 |
| 1.88 | 3.5344 | 1.37113 | 4.33590 | 6.64467 | 1.23420 | 2.65901 | 5.72865 |
| 1.89 | 3.5721 | 1.37477 | 4.34741 | 6.75127 | 1.23639 | 2.66371 | 5.73879 |
| **1.90** | 3.6100 | 1.37840 | 4.35890 | 6.85900 | 1.23856 | 2.66840 | 5.74890 |
| 1.91 | 3.6481 | 1.38203 | 4.37035 | 6.96787 | 1.24073 | 2.67307 | 5.75897 |
| 1.92 | 3.6864 | 1.38564 | 4.38178 | 7.07789 | 1.24289 | 2.67773 | 5.76900 |
| 1.93 | 3.7249 | 1.38924 | 4.39318 | 7.18906 | 1.24505 | 2.68237 | 5.77900 |
| 1.94 | 3.7636 | 1.39284 | 4.40454 | 7.30138 | 1.24719 | 2.68700 | 5.78896 |
| 1.95 | 3.8025 | 1.39642 | 4.41588 | 7.41488 | 1.24933 | 2.69161 | 5.79889 |
| 1.96 | 3.8416 | 1.40000 | 4.42719 | 7.52954 | 1.25146 | 2.69620 | 5.80879 |
| 1.97 | 3.8809 | 1.40357 | 4.43847 | 7.64537 | 1.25359 | 2.70078 | 5.81865 |
| 1.98 | 3.9204 | 1.40712 | 4.44972 | 7.76239 | 1.25571 | 2.70534 | 5.82848 |
| 1.99 | 3.9601 | 1.41067 | 4.46094 | 7.88060 | 1.25782 | 2.70989 | 5.83827 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **2.00** | 4.0000 | 1.41421 | 4.47214 | 8.00000 | 1.25992 | 2.71442 | 5.84804 |
| 2.01 | 4.0401 | 1.41774 | 4.48330 | 8.12060 | 1.26202 | 2.71893 | 5.85777 |
| 2.02 | 4.0804 | 1.42127 | 4.49444 | 8.24241 | 1.26411 | 2.72344 | 5.86746 |
| 2.03 | 4.1209 | 1.42478 | 4.50555 | 8.36543 | 1.26619 | 2.72792 | 5.87713 |
| 2.04 | 4.1616 | 1.42829 | 4.51664 | 8.48966 | 1.26827 | 2.73239 | 5.88677 |
| 2.05 | 4.2025 | 1.43178 | 4.52769 | 8.61512 | 1.27033 | 2.73685 | 5.89637 |
| 2.06 | 4.2436 | 1.43527 | 4.53872 | 8.74182 | 1.27240 | 2.74129 | 5.90594 |
| 2.07 | 4.2849 | 1.43875 | 4.54973 | 8.86974 | 1.27445 | 2.74572 | 5.91548 |
| 2.08 | 4.3264 | 1.44222 | 4.56070 | 8.99891 | 1.27650 | 2.75014 | 5.92499 |
| 2.09 | 4.3681 | 1.44568 | 4.57165 | 9.12933 | 1.27854 | 2.75454 | 5.93447 |
| **2.10** | 4.4100 | 1.44914 | 4.58258 | 9.26100 | 1.28058 | 2.75892 | 5.94392 |
| 2.11 | 4.4521 | 1.45258 | 4.59347 | 9.39393 | 1.28261 | 2.76330 | 5.95334 |
| 2.12 | 4.4944 | 1.45602 | 4.60435 | 9.52813 | 1.28463 | 2.76766 | 5.96273 |
| 2.13 | 4.5369 | 1.45945 | 4.61519 | 9.66360 | 1.28665 | 2.77200 | 5.97209 |
| 2.14 | 4.5796 | 1.46287 | 4.62601 | 9.80034 | 1.28866 | 2.77633 | 5.98142 |
| 2.15 | 4.6225 | 1.46629 | 4.63681 | 9.93838 | 1.29066 | 2.78065 | 5.99073 |
| 2.16 | 4.6656 | 1.46969 | 4.64758 | 10.0777 | 1.29266 | 2.78495 | 6.00000 |
| 2.17 | 4.7089 | 1.47309 | 4.65833 | 10.2183 | 1.29465 | 2.78924 | 6.00925 |
| 2.18 | 4.7524 | 1.47648 | 4.66905 | 10.3602 | 1.29664 | 2.79352 | 6.01846 |
| 2.19 | 4.7961 | 1.47986 | 4.67974 | 10.5035 | 1.29862 | 2.79779 | 6.02765 |
| **2.20** | 4.8400 | 1.48324 | 4.69042 | 10.6480 | 1.30059 | 2.80204 | 6.03681 |
| 2.21 | 4.8841 | 1.48661 | 4.70106 | 10.7939 | 1.30256 | 2.80628 | 6.04594 |
| 2.22 | 4.9284 | 1.48997 | 4.71169 | 10.9410 | 1.30452 | 2.81050 | 6.05505 |
| 2.23 | 4.9729 | 1.49332 | 4.72229 | 11.0896 | 1.30648 | 2.81472 | 6.06413 |
| 2.24 | 5.0176 | 1.49666 | 4.73286 | 11.2394 | 1.30843 | 2.81892 | 6.07318 |
| 2.25 | 5.0625 | 1.50000 | 4.74342 | 11.3906 | 1.31037 | 2.82311 | 6.08220 |
| 2.26 | 5.1076 | 1.50333 | 4.75395 | 11.5432 | 1.31231 | 2.82728 | 6.09120 |
| 2.27 | 5.1529 | 1.50665 | 4.76445 | 11.6971 | 1.31424 | 2.83145 | 6.10017 |
| 2.28 | 5.1984 | 1.50997 | 4.77493 | 11.8524 | 1.31617 | 2.83560 | 6.10911 |
| 2.29 | 5.2441 | 1.51327 | 4.78539 | 12.0090 | 1.31809 | 2.83974 | 6.11803 |
| **2.30** | 5.2900 | 1.51658 | 4.79583 | 12.1670 | 1.32001 | 2.84387 | 6.12693 |
| 2.31 | 5.3361 | 1.51987 | 4.80625 | 12.3264 | 1.32192 | 2.84798 | 6.13579 |
| 2.32 | 5.3824 | 1.52315 | 4.81664 | 12.4872 | 1.32382 | 2.85209 | 6.14463 |
| 2.33 | 5.4289 | 1.52643 | 4.82701 | 12.6493 | 1.32572 | 2.85618 | 6.15345 |
| 2.34 | 5.4756 | 1.52971 | 4.83735 | 12.8129 | 1.32761 | 2.86026 | 6.16224 |
| 2.35 | 5.5225 | 1.53297 | 4.84768 | 12.9779 | 1.32950 | 2.86433 | 6.17101 |
| 2.36 | 5.5696 | 1.53623 | 4.85798 | 13.1443 | 1.33139 | 2.86838 | 6.17975 |
| 2.37 | 5.6169 | 1.53948 | 4.86826 | 13.3121 | 1.33326 | 2.87243 | 6.18846 |
| 2.38 | 5.6644 | 1.54272 | 4.87852 | 13.4813 | 1.33514 | 2.87646 | 6.19715 |
| 2.39 | 5.7121 | 1.54596 | 4.88876 | 13.6519 | 1.33700 | 2.88049 | 6.20582 |
| **2.40** | 5.7600 | 1.54919 | 4.89898 | 13.8240 | 1.33887 | 2.88450 | 6.21447 |
| 2.41 | 5.8081 | 1.55242 | 4.90918 | 13.9975 | 1.34072 | 2.88850 | 6.22308 |
| 2.42 | 5.8564 | 1.55563 | 4.91935 | 14.1725 | 1.34257 | 2.89249 | 6.23168 |
| 2.43 | 5.9049 | 1.55885 | 4.92950 | 14.3489 | 1.34442 | 2.89647 | 6.24025 |
| 2.44 | 5.9536 | 1.56205 | 4.93964 | 14.5268 | 1.34626 | 2.90044 | 6.24880 |
| 2.45 | 6.0025 | 1.56525 | 4.94975 | 14.7061 | 1.34810 | 2.90439 | 6.25732 |
| 2.46 | 6.0516 | 1.56844 | 4.95984 | 14.8869 | 1.34993 | 2.90834 | 6.26583 |
| 2.47 | 6.1009 | 1.57162 | 4.96991 | 15.0692 | 1.35176 | 2.91227 | 6.27431 |
| 2.48 | 6.1504 | 1.57480 | 4.97996 | 15.2530 | 1.35358 | 2.91620 | 6.28276 |
| 2.49 | 6.2001 | 1.57797 | 4.98999 | 15.4382 | 1.35540 | 2.92011 | 6.29119 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **2.50** | 6.2500 | 1.58114 | 5.00000 | 15.6250 | 1.35721 | 2.92402 | 6.29961 |
| 2.51 | 6.3001 | 1.58430 | 5.00999 | 15.8133 | 1.35902 | 2.92791 | 6.30799 |
| 2.52 | 6.3504 | 1.58745 | 5.01996 | 16.0030 | 1.36082 | 2.93179 | 6.31636 |
| 2.53 | 6.4009 | 1.59060 | 5.02991 | 16.1943 | 1.36262 | 2.93567 | 6.32470 |
| 2.54 | 6.4516 | 1.59374 | 5.03984 | 16.3871 | 1.36441 | 2.93953 | 6.33303 |
| 2.55 | 6.5025 | 1.59687 | 5.04975 | 16.5814 | 1.36620 | 2.94338 | 6.34133 |
| 2.56 | 6.5536 | 1.60000 | 5.05964 | 16.7772 | 1.36798 | 2.94723 | 6.34960 |
| 2.57 | 6.6049 | 1.60312 | 5.06952 | 16.9746 | 1.36976 | 2.95106 | 6.35786 |
| 2.58 | 6.6564 | 1.60624 | 5.07937 | 17.1735 | 1.37153 | 2.95488 | 6.36610 |
| 2.59 | 6.7081 | 1.60935 | 5.08920 | 17.3740 | 1.37330 | 2.95869 | 6.37431 |
| **2.60** | 6.7600 | 1.61245 | 5.09902 | 17.5760 | 1.37507 | 2.96250 | 6.38250 |
| 2.61 | 6.8121 | 1.61555 | 5.10882 | 17.7796 | 1.37683 | 2.96629 | 6.39068 |
| 2.62 | 6.8644 | 1.61864 | 5.11859 | 17.9847 | 1.37859 | 2.97007 | 6.39883 |
| 2.63 | 6.9169 | 1.62173 | 5.12835 | 18.1914 | 1.38034 | 2.97385 | 6.40696 |
| 2.64 | 6.9696 | 1.62481 | 5.13809 | 18.3997 | 1.38208 | 2.97761 | 6.41507 |
| 2.65 | 7.0225 | 1.62788 | 5.14782 | 18.6096 | 1.38383 | 2.98137 | 6.42316 |
| 2.66 | 7.0756 | 1.63095 | 5.15752 | 18.8211 | 1.38557 | 2.98511 | 6.43123 |
| 2.67 | 7.1289 | 1.63401 | 5.16720 | 19.0342 | 1.38730 | 2.98885 | 6.43928 |
| 2.68 | 7.1824 | 1.63707 | 5.17687 | 19.2488 | 1.38903 | 2.99257 | 6.44731 |
| 2.69 | 7.2361 | 1.64012 | 5.18652 | 19.4651 | 1.39076 | 2.99629 | 6.45531 |
| **2.70** | 7.2900 | 1.64317 | 5.19615 | 19.6830 | 1.39248 | 3.00000 | 6.46330 |
| 2.71 | 7.3441 | 1.64621 | 5.20577 | 19.9025 | 1.39419 | 3.00370 | 6.47127 |
| 2.72 | 7.3984 | 1.64924 | 5.21536 | 20.1236 | 1.39591 | 3.00739 | 6.47922 |
| 2.73 | 7.4529 | 1.65227 | 5.22494 | 20.3464 | 1.39761 | 3.01107 | 6.48715 |
| 2.74 | 7.5076 | 1.65529 | 5.23450 | 20.5708 | 1.39932 | 3.01474 | 6.49507 |
| 2.75 | 7.5625 | 1.65831 | 5.24404 | 20.7969 | 1.40102 | 3.01841 | 6.50296 |
| 2.76 | 7.6176 | 1.66132 | 5.25357 | 21.0246 | 1.40272 | 3.02206 | 6.51083 |
| 2.77 | 7.6729 | 1.66433 | 5.26308 | 21.2539 | 1.40441 | 3.02570 | 6.51868 |
| 2.78 | 7.7284 | 1.66733 | 5.27257 | 21.4850 | 1.40610 | 3.02934 | 6.52652 |
| 2.79 | 7.7841 | 1.67033 | 5.28205 | 21.7176 | 1.40778 | 3.03297 | 6.53434 |
| **2.80** | 7.8400 | 1.67332 | 5.29150 | 21.9520 | 1.40946 | 3.03659 | 6.54213 |
| 2.81 | 7.8961 | 1.67631 | 5.30094 | 22.1880 | 1.41114 | 3.04020 | 6.54991 |
| 2.82 | 7.9524 | 1.67929 | 5.31037 | 22.4258 | 1.41281 | 3.04380 | 6.55767 |
| 2.83 | 8.0089 | 1.68226 | 5.31977 | 22.6652 | 1.41448 | 3.04740 | 6.56541 |
| 2.84 | 8.0656 | 1.68523 | 5.32917 | 22.9063 | 1.41614 | 3.05098 | 6.57314 |
| 2.85 | 8.1225 | 1.68819 | 5.33854 | 23.1491 | 1.41780 | 3.05456 | 6.58084 |
| 2.86 | 8.1796 | 1.69115 | 5.34790 | 23.3937 | 1.41946 | 3.05813 | 6.58853 |
| 2.87 | 8.2369 | 1.69411 | 5.35724 | 23.6399 | 1.42111 | 3.06169 | 6.59620 |
| 2.88 | 8.2944 | 1.69706 | 5.36656 | 23.8879 | 1.42276 | 3.06524 | 6.60385 |
| 2.89 | 8.3521 | 1.70000 | 5.37587 | 24.1376 | 1.42440 | 3.06878 | 6.61149 |
| **2.90** | 8.4100 | 1.70294 | 5.38516 | 24.3890 | 1.42604 | 3.07232 | 6.61911 |
| 2.91 | 8.4681 | 1.70587 | 5.39444 | 24.6422 | 1.42768 | 3.07584 | 6.62671 |
| 2.92 | 8.5264 | 1.70880 | 5.40370 | 24.8971 | 1.42931 | 3.07936 | 6.63429 |
| 2.93 | 8.5849 | 1.71172 | 5.41295 | 25.1538 | 1.43094 | 3.08287 | 6.64185 |
| 2.94 | 8.6436 | 1.71464 | 5.42218 | 25.4122 | 1.43257 | 3.08638 | 6.64940 |
| 2.95 | 8.7025 | 1.71756 | 5.43139 | 25.6724 | 1.43419 | 3.08987 | 6.65693 |
| 2.96 | 8.7616 | 1.72047 | 5.44059 | 25.9343 | 1.43581 | 3.09336 | 6.66444 |
| 2.97 | 8.8209 | 1.72337 | 5.44977 | 26.1981 | 1.43743 | 3.09684 | 6.67194 |
| 2.98 | 8.8804 | 1.72627 | 5.45894 | 26.4636 | 1.43904 | 3.10031 | 6.67942 |
| 2.99 | 8.9401 | 1.72916 | 5.46809 | 26.7309 | 1.44065 | 3.10378 | 6.68688 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **3.00** | 9.0000 | 1.73205 | 5.47723 | 27.0000 | 1.44225 | 3.10723 | 6.69433 |
| 3.01 | 9.0601 | 1.73494 | 5.48635 | 27.2709 | 1.44385 | 3.11068 | 6.70176 |
| 3.02 | 9.1204 | 1.73781 | 5.49545 | 27.5436 | 1.44545 | 3.11412 | 6.70917 |
| 3.03 | 9.1809 | 1.74069 | 5.50454 | 27.8181 | 1.44704 | 3.11756 | 6.71657 |
| 3.04 | 9.2416 | 1.74356 | 5.51362 | 28.0945 | 1.44863 | 3.12098 | 6.72395 |
| 3.05 | 9.3025 | 1.74642 | 5.52268 | 28.3726 | 1.45022 | 3.12440 | 6.73132 |
| 3.06 | 9.3636 | 1.74929 | 5.53173 | 28.6526 | 1.45180 | 3.12781 | 6.73866 |
| 3.07 | 9.4249 | 1.75214 | 5.54076 | 28.9344 | 1.45338 | 3.13121 | 6.74600 |
| 3.08 | 9.4864 | 1.75499 | 5.54977 | 29.2181 | 1.45496 | 3.13461 | 6.75331 |
| 3.09 | 9.5481 | 1.75784 | 5.55878 | 29.5036 | 1.45653 | 3.13800 | 6.76061 |
| **3.10** | 9.6100 | 1.76068 | 5.56776 | 29.7910 | 1.45810 | 3.14138 | 6.76790 |
| 3.11 | 9.6721 | 1.76352 | 5.57674 | 30.0802 | 1.45967 | 3.14475 | 6.77517 |
| 3.12 | 9.7344 | 1.76635 | 5.58570 | 30.3713 | 1.46123 | 3.14812 | 6.78242 |
| 3.13 | 9.7969 | 1.76918 | 5.59464 | 30.6643 | 1.46279 | 3.15148 | 6.78966 |
| 3.14 | 9.8596 | 1.77200 | 5.60357 | 30.9591 | 1.46434 | 3.15483 | 6.79688 |
| 3.15 | 9.9225 | 1.77482 | 5.61249 | 31.2559 | 1.46590 | 3.15818 | 6.80409 |
| 3.16 | 9.9856 | 1.77764 | 5.62139 | 31.5545 | 1.46745 | 3.16152 | 6.81128 |
| 3.17 | 10.0489 | 1.78045 | 5.63028 | 31.8550 | 1.46899 | 3.16485 | 6.81846 |
| 3.18 | 10.1124 | 1.78326 | 5.63915 | 32.1574 | 1.47054 | 3.16817 | 6.82562 |
| 3.19 | 10.1761 | 1.78606 | 5.64801 | 32.4618 | 1.47208 | 3.17149 | 6.83277 |
| **3.20** | 10.2400 | 1.78885 | 5.65685 | 32.7680 | 1.47361 | 3.17480 | 6.83990 |
| 3.21 | 10.3041 | 1.79165 | 5.66569 | 33.0762 | 1.47515 | 3.17811 | 6.84702 |
| 3.22 | 10.3684 | 1.79444 | 5.67450 | 33.3862 | 1.47668 | 3.18140 | 6.85412 |
| 3.23 | 10.4329 | 1.79722 | 5.68331 | 33.6983 | 1.47820 | 3.18469 | 6.86121 |
| 3.24 | 10.4976 | 1.80000 | 5.69210 | 34.0122 | 1.47973 | 3.18798 | 6.86829 |
| 3.25 | 10.5625 | 1.80278 | 5.70088 | 34.3281 | 1.48125 | 3.19125 | 6.87534 |
| 3.26 | 10.6276 | 1.80555 | 5.70964 | 34.6460 | 1.48277 | 3.19452 | 6.88239 |
| 3.27 | 10.6929 | 1.80831 | 5.71839 | 34.9658 | 1.48428 | 3.19778 | 6.88942 |
| 3.28 | 10.7584 | 1.81108 | 5.72713 | 35.2876 | 1.48579 | 3.20104 | 6.89643 |
| 3.29 | 10.8241 | 1.81384 | 5.73585 | 35.6113 | 1.48730 | 3.20429 | 6.90344 |
| **3.30** | 10.8900 | 1.81659 | 5.74456 | 35.9370 | 1.48881 | 3.20753 | 6.91042 |
| 3.31 | 10.9561 | 1.81934 | 5.75326 | 36.2647 | 1.49031 | 3.21077 | 6.91740 |
| 3.32 | 11.0224 | 1.82209 | 5.76194 | 36.5944 | 1.49181 | 3.21400 | 6.92436 |
| 3.33 | 11.0889 | 1.82483 | 5.77062 | 36.9260 | 1.49330 | 3.21722 | 6.93130 |
| 3.34 | 11.1556 | 1.82757 | 5.77927 | 37.2597 | 1.49480 | 3.22044 | 6.93823 |
| 3.35 | 11.2225 | 1.83030 | 5.78792 | 37.5954 | 1.49629 | 3.22365 | 6.94515 |
| 3.36 | 11.2896 | 1.83303 | 5.79655 | 37.9331 | 1.49777 | 3.22686 | 6.95205 |
| 3.37 | 11.3569 | 1.83576 | 5.80517 | 38.2728 | 1.49926 | 3.23006 | 6.95894 |
| 3.38 | 11.4244 | 1.83848 | 5.81378 | 38.6145 | 1.50074 | 3.23325 | 6.96582 |
| 3.39 | 11.4921 | 1.84120 | 5.82237 | 38.9582 | 1.50222 | 3.23643 | 6.97268 |
| **3.40** | 11.5600 | 1.84391 | 5.83095 | 39.3040 | 1.50369 | 3.23961 | 6.97953 |
| 3.41 | 11.6281 | 1.84662 | 5.83952 | 39.6518 | 1.50517 | 3.24278 | 6.98637 |
| 3.42 | 11.6964 | 1.84932 | 5.84808 | 40.0017 | 1.50664 | 3.24595 | 6.99319 |
| 3.43 | 11.7649 | 1.85203 | 5.85662 | 40.3536 | 1.50810 | 3.24911 | 7.00000 |
| 3.44 | 11.8336 | 1.85472 | 5.86515 | 40.7076 | 1.50957 | 3.25227 | 7.00680 |
| 3.45 | 11.9025 | 1.85742 | 5.87367 | 41.0636 | 1.51103 | 3.25542 | 7.01358 |
| 3.46 | 11.9716 | 1.86011 | 5.88218 | 41.4217 | 1.51249 | 3.25856 | 7.02035 |
| 3.47 | 12.0409 | 1.86279 | 5.89067 | 41.7819 | 1.51394 | 3.26169 | 7.02711 |
| 3.48 | 12.1104 | 1.86548 | 5.89915 | 42.1442 | 1.51540 | 3.26482 | 7.03385 |
| 3.49 | 12.1801 | 1.86815 | 5.90762 | 42.5085 | 1.51685 | 3.26795 | 7.04058 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **3.50** | 12.2500 | 1.87083 | 5.91608 | 42.8750 | 1.51829 | 3.27107 | 7.04730 |
| 3.51 | 12.3201 | 1.87350 | 5.92453 | 43.2436 | 1.51974 | 3.27418 | 7.05400 |
| 3.52 | 12.3904 | 1.87617 | 5.93296 | 43.6142 | 1.52118 | 3.27729 | 7.06070 |
| 3.53 | 12.4609 | 1.87883 | 5.94138 | 43.9870 | 1.52262 | 3.28039 | 7.06738 |
| 3.54 | 12.5316 | 1.88149 | 5.94979 | 44.3619 | 1.52406 | 3.28348 | 7.07404 |
| 3.55 | 12.6025 | 1.88414 | 5.95819 | 44.7389 | 1.52549 | 3.28657 | 7.08070 |
| 3.56 | 12.6736 | 1.88680 | 5.96657 | 45.1180 | 1.52692 | 3.28965 | 7.08734 |
| 3.57 | 12.7449 | 1.88944 | 5.97495 | 45.4993 | 1.52835 | 3.29273 | 7.09397 |
| 3.58 | 12.8164 | 1.89209 | 5.98331 | 45.8827 | 1.52978 | 3.29580 | 7.10059 |
| 3.59 | 12.8881 | 1.89473 | 5.99166 | 46.2683 | 1.53120 | 3.29887 | 7.10719 |
| **3.60** | 12.9600 | 1.89737 | 6.00000 | 46.6560 | 1.53262 | 3.30193 | 7.11379 |
| 3.61 | 13.0321 | 1.90000 | 6.00833 | 47.0459 | 1.53404 | 3.30498 | 7.12037 |
| 3.62 | 13.1044 | 1.90263 | 6.01664 | 47.4379 | 1.53545 | 3.30803 | 7.12694 |
| 3.63 | 13.1769 | 1.90526 | 6.02495 | 47.8321 | 1.53686 | 3.31107 | 7.13349 |
| 3.64 | 13.2496 | 1.90788 | 6.03324 | 48.2285 | 1.53827 | 3.31411 | 7.14004 |
| 3.65 | 13.3225 | 1.91050 | 6.04152 | 48.6271 | 1.53968 | 3.31714 | 7.14657 |
| 3.66 | 13.3956 | 1.91311 | 6.04979 | 49.0279 | 1.54109 | 3.32017 | 7.15309 |
| 3.67 | 13.4689 | 1.91572 | 6.05805 | 49.4309 | 1.54249 | 3.32319 | 7.15960 |
| 3.68 | 13.5424 | 1.91833 | 6.06630 | 49.8360 | 1.54389 | 3.32621 | 7.16610 |
| 3.69 | 13.6161 | 1.92094 | 6.07454 | 50.2434 | 1.54529 | 3.32922 | 7.17258 |
| **3.70** | 13.6900 | 1.92354 | 6.08276 | 50.6530 | 1.54668 | 3.33222 | 7.17905 |
| 3.71 | 13.7641 | 1.92614 | 6.09098 | 51.0648 | 1.54807 | 3.33522 | 7.18552 |
| 3.72 | 13.8384 | 1.92873 | 6.09918 | 51.4788 | 1.54946 | 3.33822 | 7.19197 |
| 3.73 | 13.9129 | 1.93132 | 6.10737 | 51.8951 | 1.55085 | 3.34120 | 7.19840 |
| 3.74 | 13.9876 | 1.93391 | 6.11555 | 52.3136 | 1.55223 | 3.34419 | 7.20483 |
| 3.75 | 14.0625 | 1.93649 | 6.12372 | 52.7344 | 1.55362 | 3.34716 | 7.21125 |
| 3.76 | 14.1376 | 1.93907 | 6.13188 | 53.1574 | 1.55500 | 3.35014 | 7.21765 |
| 3.77 | 14.2129 | 1.94165 | 6.14003 | 53.5826 | 1.55637 | 3.35310 | 7.22405 |
| 3.78 | 14.2884 | 1.94422 | 6.14817 | 54.0102 | 1.55775 | 3.35607 | 7.23043 |
| 3.79 | 14.3641 | 1.94679 | 6.15630 | 54.4399 | 1.55912 | 3.35902 | 7.23680 |
| **3.80** | 14.4400 | 1.94936 | 6.16441 | 54.8720 | 1.56049 | 3.36198 | 7.24316 |
| 3.81 | 14.5161 | 1.95192 | 6.17252 | 55.3063 | 1.56186 | 3.36492 | 7.24950 |
| 3.82 | 14.5924 | 1.95448 | 6.18061 | 55.7430 | 1.56322 | 3.36786 | 7.25584 |
| 3.83 | 14.6689 | 1.95704 | 6.18870 | 56.1819 | 1.56459 | 3.37080 | 7.26217 |
| 3.84 | 14.7456 | 1.95959 | 6.19677 | 56.6231 | 1.56595 | 3.37373 | 7.26848 |
| 3.85 | 14.8225 | 1.96214 | 6.20484 | 57.0666 | 1.56731 | 3.37666 | 7.27479 |
| 3.86 | 14.8996 | 1.96469 | 6.21289 | 57.5125 | 1.56866 | 3.37958 | 7.28108 |
| 3.87 | 14.9769 | 1.96723 | 6.22093 | 57.9606 | 1.57001 | 3.38249 | 7.28736 |
| 3.88 | 15.0544 | 1.96977 | 6.22896 | 58.4111 | 1.57137 | 3.38540 | 7.29363 |
| 3.89 | 15.1321 | 1.97231 | 6.23699 | 58.8639 | 1.57271 | 3.38831 | 7.29989 |
| **3.90** | 15.2100 | 1.97484 | 6.24500 | 59.3190 | 1.57406 | 3.39121 | 7.30614 |
| 3.91 | 15.2881 | 1.97737 | 6.25300 | 59.7765 | 1.57541 | 3.39411 | 7.31238 |
| 3.92 | 15.3664 | 1.97990 | 6.26099 | 60.2363 | 1.57675 | 3.39700 | 7.31861 |
| 3.93 | 15.4449 | 1.98242 | 6.26897 | 60.6985 | 1.57809 | 3.39988 | 7.32483 |
| 3.94 | 15.5236 | 1.98494 | 6.27694 | 61.1630 | 1.57942 | 3.40277 | 7.33104 |
| 3.95 | 15.6025 | 1.98746 | 6.28490 | 61.6299 | 1.58076 | 3.40564 | 7 33723 |
| 3.96 | 15.6816 | 1.98997 | 6.29285 | 62.0991 | 1.58209 | 3.40851 | 7.34342 |
| 3.97 | 15.7609 | 1.99249 | 6.30079 | 62.5708 | 1.58342 | 3.41138 | 7.34960 |
| 3.98 | 15.8404 | 1.99499 | 6.30872 | 63.0448 | 1.58475 | 3.41424 | 7.35576 |
| 3.99 | 15.9201 | 1.99750 | 6.31664 | 63 5212 | 1.58608 | 3.41710 | 7.36192 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **4.00** | 16.0000 | 2.00000 | 6.32456 | 64.0000 | 1.58740 | 3.41995 | 7.36806 |
| 4.01 | 16.0801 | 2.00250 | 6.33246 | 64.4812 | 1.58872 | 3.42280 | 7.37420 |
| 4.02 | 16.1604 | 2.00499 | 6.34035 | 64.9648 | 1.59004 | 3.42564 | 7.38032 |
| 4.03 | 16.2409 | 2.00749 | 6.34823 | 65.4508 | 1.59136 | 3.42848 | 7.38644 |
| 4.04 | 16.3216 | 2.00998 | 6.35610 | 65.9393 | 1.59267 | 3.43131 | 7.39254 |
| 4.05 | 16.4025 | 2.01246 | 6.36396 | 66.4301 | 1.59399 | 3.43414 | 7.39864 |
| 4.06 | 16.4836 | 2.01494 | 6.37181 | 66.9234 | 1.59530 | 3.43697 | 7.40472 |
| 4.07 | 16.5649 | 2.01742 | 6.37966 | 67.4191 | 1.59661 | 3.43979 | 7.41080 |
| 4.08 | 16.6464 | 2.01990 | 6.38749 | 67.9173 | 1.59791 | 3.44260 | 7.41686 |
| 4.09 | 16.7281 | 2.02237 | 6.39531 | 68.4179 | 1.59922 | 3.44541 | 7.42291 |
| **4.10** | 16.8100 | 2.02485 | 6.40312 | 68.9210 | 1.60052 | 3.44822 | 7.42896 |
| 4.11 | 16.8921 | 2.02731 | 6.41093 | 69.4265 | 1.60182 | 3.45102 | 7.43499 |
| 4.12 | 16.9744 | 2.02978 | 6.41872 | 69.9345 | 1.60312 | 3.45382 | 7.44102 |
| 4.13 | 17.0569 | 2.03224 | 6.42651 | 70.4450 | 1.60441 | 3.45661 | 7.44703 |
| 4.14 | 17.1396 | 2.03470 | 6.43428 | 70.9579 | 1.60571 | 3.45939 | 7.45304 |
| 4.15 | 17.2225 | 2.03715 | 6.44205 | 71.4734 | 1.60700 | 3.46218 | 7.45904 |
| 4.16 | 17.3056 | 2.03961 | 6.44981 | 71.9913 | 1.60829 | 3.46496 | 7.46502 |
| 4.17 | 17.3889 | 2.04206 | 6.45755 | 72.5117 | 1.60958 | 3.46773 | 7.47100 |
| 4.18 | 17.4724 | 2.04450 | 6.46529 | 73.0346 | 1.61086 | 3.47050 | 7.47697 |
| 4.19 | 17.5561 | 2.04695 | 6.47302 | 73.5601 | 1.61215 | 3.47327 | 7.48292 |
| **4.20** | 17.6400 | 2.04939 | 6.48074 | 74.0880 | 1.61343 | 3.47603 | 7.48887 |
| 4.21 | 17.7241 | 2.05183 | 6.48845 | 74.6185 | 1.61471 | 3.47878 | 7.49481 |
| 4.22 | 17.8084 | 2.05426 | 6.49615 | 75.1514 | 1.61599 | 3.48154 | 7.50074 |
| 4.23 | 17.8929 | 2.05670 | 6.50384 | 75.6870 | 1.61726 | 3.48428 | 7.50666 |
| 4.24 | 17.9776 | 2.05913 | 6.51153 | 76.2250 | 1.61853 | 3.48703 | 7.51257 |
| 4.25 | 18.0625 | 2.06155 | 6.51920 | 76.7656 | 1.61981 | 3.48977 | 7.51847 |
| 4.26 | 18.1476 | 2.06398 | 6.52687 | 77.3088 | 1.62108 | 3.49250 | 7.52437 |
| 4.27 | 18.2329 | 2.06640 | 6.53452 | 77.8545 | 1.62234 | 3.49523 | 7.53025 |
| 4.28 | 18.3184 | 2.06882 | 6.54217 | 78.4028 | 1.62361 | 3.49796 | 7.53612 |
| 4.29 | 18.4041 | 2.07123 | 6.54981 | 78.9536 | 1.62487 | 3.50068 | 7.54199 |
| **4.30** | 18.4900 | 2.07364 | 6.55744 | 79.5070 | 1.62613 | 3.50340 | 7.54784 |
| 4.31 | 18.5761 | 2.07605 | 6.56506 | 80.0630 | 1.62739 | 3.50611 | 7.55369 |
| 4.32 | 18.6624 | 2.07846 | 6.57267 | 80.6216 | 1.62865 | 3.50882 | 7.55953 |
| 4.33 | 18.7489 | 2.08087 | 6.58027 | 81.1827 | 1.62991 | 3.51153 | 7.56535 |
| 4.34 | 18.8356 | 2.08327 | 6.58787 | 81.7465 | 1.63116 | 3.51423 | 7.57117 |
| 4.35 | 18.9225 | 2.08567 | 6.59545 | 82.3129 | 1.63241 | 3.51692 | 7.57698 |
| 4.36 | 19.0096 | 2.08806 | 6.60303 | 82.8819 | 1.63366 | 3.51962 | 7.58279 |
| 4.37 | 19.0969 | 2.09045 | 6.61060 | 83.4535 | 1.63491 | 3.52231 | 7.58858 |
| 4.38 | 19.1844 | 2.09284 | 6.61816 | 84.0277 | 1.63619 | 3.52499 | 7.59436 |
| 4.39 | 19.2721 | 2.09523 | 6.62571 | 84.6045 | 1.63740 | 3.52767 | 7.60014 |
| **4.40** | 19.3600 | 2.09762 | 6.63325 | 85.1840 | 1.63864 | 3.53035 | 7.60590 |
| 4.41 | 19.4481 | 2.10000 | 6.64078 | 85.7661 | 1.63988 | 3.53302 | 7.61166 |
| 4.42 | 19.5364 | 2.10238 | 6.64831 | 86.3509 | 1.64112 | 3.53569 | 7.61741 |
| 4.43 | 19.6249 | 2.10476 | 6.65582 | 86.9383 | 1.64236 | 3.53835 | 7.62315 |
| 4.44 | 19.7136 | 2.10713 | 6.66333 | 87.5284 | 1.64359 | 3.54101 | 7.62888 |
| 4.45 | 19.8025 | 2.10950 | 6.67083 | 88.1211 | 1.64483 | 3.54367 | 7.63461 |
| 4.46 | 19.8916 | 2.11187 | 6.67832 | 88.7165 | 1.64606 | 3.54632 | 7.64032 |
| 4.47 | 19.9809 | 2.11424 | 6.68581 | 89.3146 | 1.64729 | 3.54897 | 7.64603 |
| 4.48 | 20.0704 | 2.11660 | 6.69328 | 89.9154 | 1.64851 | 3.55162 | 7.65172 |
| 4.49 | 20.1601 | 2.11896 | 6.70075 | 90.5188 | 1.64974 | 3.55428 | 7.65741 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **4.50** | 20.2500 | 2.12132 | 6.70820 | 91.1250 | 1.65096 | 3.55689 | 7.66309 |
| 4.51 | 20.3401 | 2.12368 | 6.71565 | 91.7339 | 1.65219 | 3.55953 | 7.66877 |
| 4.52 | 20.4304 | 2.12603 | 6.72309 | 92.3454 | 1.65341 | 3.56215 | 7.67443 |
| 4.53 | 20.5209 | 2.12838 | 6.73053 | 92.9597 | 1.65462 | 3.56478 | 7.68009 |
| 4.54 | 20.6116 | 2.13073 | 6.73795 | 93.5767 | 1.65584 | 3.56740 | 7.68573 |
| 4.55 | 20.7025 | 2.13307 | 6.74537 | 94.1964 | 1.65706 | 3.57002 | 7.69137 |
| 4.56 | 20.7936 | 2.13542 | 6.75278 | 94.8188 | 1.65827 | 3.57263 | 7.69700 |
| 4.57 | 20.8849 | 2.13776 | 6.76018 | 95.4440 | 1.65948 | 3.57524 | 7.70262 |
| 4.58 | 20.9764 | 2.14009 | 6.76757 | 96.0719 | 1.66069 | 3.57785 | 7.70824 |
| 4.59 | 21.0681 | 2.14243 | 6.77495 | 96.7026 | 1.66190 | 3.58045 | 7.71384 |
| **4.60** | 21.1600 | 2.14476 | 6.78233 | 97.3360 | 1.66310 | 3.58305 | 7.71944 |
| 4.61 | 21.2521 | 2.14709 | 6.78970 | 97.9722 | 1.66431 | 3.58564 | 7.72503 |
| 4.62 | 21.3444 | 2.14942 | 6.79706 | 98.6111 | 1.66551 | 3.58823 | 7.73061 |
| 4.63 | 21.4369 | 2.15174 | 6.80441 | 99.2528 | 1.66671 | 3.59082 | 7.73619 |
| 4.64 | 21.5296 | 2.15407 | 6.81175 | 99.8973 | 1.66791 | 3.59340 | 7.74175 |
| 4.65 | 21.6225 | 2.15639 | 6.81909 | 100.545 | 1.66911 | 3.59598 | 7.74731 |
| 4.66 | 21.7156 | 2.15870 | 6.82642 | 101.195 | 1.67030 | 3.59856 | 7.75286 |
| 4.67 | 21.8089 | 2.16102 | 6.83374 | 101.848 | 1.67150 | 3.60113 | 7.75840 |
| 4.68 | 21.9024 | 2.16333 | 6.84105 | 102.503 | 1.67269 | 3.60370 | 7.76394 |
| 4.69 | 21.9961 | 2.16564 | 6.84836 | 103.162 | 1.67388 | 3.60626 | 7.76946 |
| **4.70** | 22.0900 | 2.16795 | 6.85565 | 103.823 | 1.67507 | 3.60883 | 7.77498 |
| 4.71 | 22.1841 | 2.17025 | 6.86294 | 104.487 | 1.67626 | 3.61138 | 7.78049 |
| 4.72 | 22.2784 | 2.17256 | 6.87023 | 105.154 | 1.67744 | 3.61394 | 7.78599 |
| 4.73 | 22.3729 | 2.17486 | 6.87750 | 105.824 | 1.67863 | 3.61649 | 7.79149 |
| 4.74 | 22.4676 | 2.17715 | 6.88477 | 106.496 | 1.67981 | 3.61903 | 7.79697 |
| 4.75 | 22.5625 | 2.17945 | 6.89202 | 107.172 | 1.68099 | 3.62158 | 7.80245 |
| 4.76 | 22.6576 | 2.18174 | 6.89928 | 107.850 | 1.68217 | 3.62412 | 7.80793 |
| 4.77 | 22.7529 | 2.18403 | 6.90652 | 108.531 | 1.68334 | 3.62665 | 7.81339 |
| 4.78 | 22.8484 | 2.18632 | 6.91375 | 109.215 | 1.68452 | 3.62919 | 7.81885 |
| 4.79 | 22.9441 | 2.18861 | 6.92098 | 109.902 | 1.68569 | 3.63172 | 7.82429 |
| **4.80** | 23.0400 | 2.19089 | 6.92820 | 110.592 | 1.68687 | 3.63424 | 7.82974 |
| 4.81 | 23.1361 | 2.19317 | 6.93542 | 111.285 | 1.68804 | 3.63676 | 7.83517 |
| 4.82 | 23.2324 | 2.19545 | 6.94262 | 111.980 | 1.68920 | 3.63928 | 7.84059 |
| 4.83 | 23.3289 | 2.19773 | 6.94982 | 112.679 | 1.69037 | 3.64180 | 7.84601 |
| 4.84 | 23.4256 | 2.20000 | 6.95701 | 113.380 | 1.69154 | 3.64431 | 7.85142 |
| 4.85 | 23.5225 | 2.20227 | 6.96419 | 114.084 | 1.69270 | 3.64682 | 7.85683 |
| 4.86 | 23.6196 | 2.20454 | 6.97137 | 114.791 | 1.69386 | 3.64932 | 7.86222 |
| 4.87 | 23.7169 | 2.20681 | 6.97854 | 115.501 | 1.69503 | 3.65182 | 7.86761 |
| 4.88 | 23.8144 | 2.20907 | 6.98570 | 116.214 | 1.69619 | 3.65432 | 7.87299 |
| 4.89 | 23.9121 | 2.21133 | 6.99285 | 116.930 | 1.69734 | 3.65681 | 7.87837 |
| **4.90** | 24.0100 | 2.21359 | 7.00000 | 117.649 | 1.69850 | 3.65931 | 7.88374 |
| 4.91 | 24.1081 | 2.21585 | 7.00714 | 118.371 | 1.69965 | 3.66179 | 7.88909 |
| 4.92 | 24.2064 | 2.21811 | 7.01427 | 119.095 | 1.70081 | 3.66428 | 7.89445 |
| 4.93 | 24.3049 | 2.22036 | 7.02140 | 119.823 | 1.70196 | 3.66676 | 7.89979 |
| 4.94 | 24.4036 | 2.22261 | 7.02851 | 120.554 | 1.70311 | 3.66924 | 7.90513 |
| 4.95 | 24.5025 | 2.22486 | 7.03562 | 121.287 | 1.70426 | 3.67171 | 7.91046 |
| 4.96 | 24.6016 | 2.22711 | 7.04273 | 122.024 | 1.70540 | 3.67418 | 7.91578 |
| 4.97 | 24.7009 | 2.22935 | 7.04982 | 122.763 | 1.70655 | 3.67665 | 7.92110 |
| 4.98 | 24.8004 | 2.23159 | 7.05691 | 123.506 | 1.70769 | 3.67911 | 7.92641 |
| 4.99 | 24.9001 | 2.23383 | 7.06399 | 124.251 | 1.70884 | 3.68157 | 7.93171 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **5.00** | 25.0000 | 2.23607 | 7.07107 | 125.000 | 1.70998 | 3.68403 | 7.93701 |
| 5.01 | 25.1001 | 2.23830 | 7.07814 | 125.752 | 1.71112 | 3.68649 | 7.94229 |
| 5.02 | 25.2004 | 2.24054 | 7.08520 | 126.506 | 1.71225 | 3.68894 | 7.94757 |
| 5.03 | 25.3009 | 2.24277 | 7.09225 | 127.264 | 1.71339 | 3.69138 | 7.95285 |
| 5.04 | 25.4016 | 2.24499 | 7.09930 | 128.024 | 1.71452 | 3.69383 | 7.95811 |
| 5.05 | 25.5025 | 2.24722 | 7.10634 | 128.788 | 1.71566 | 3.69627 | 7.96337 |
| 5.06 | 25.6036 | 2.24944 | 7.11337 | 129.554 | 1.71679 | 3.69871 | 7.96863 |
| 5.07 | 25.7049 | 2.25167 | 7.12039 | 130.324 | 1.71792 | 3.70114 | 7.97387 |
| 5.08 | 25.8064 | 2.25389 | 7.12741 | 131.097 | 1.71905 | 3.70357 | 7.97911 |
| 5.09 | 25.9081 | 2.25610 | 7.13442 | 131.872 | 1.72017 | 3.70600 | 7.98434 |
| **5.10** | 26.0100 | 2.25832 | 7.14143 | 132.651 | 1.72130 | 3.70843 | 7.98957 |
| 5.11 | 26.1121 | 2.26053 | 7.14843 | 133.433 | 1.72242 | 3.71085 | 7.99479 |
| 5.12 | 26.2144 | 2.26274 | 7.15542 | 134.218 | 1.72355 | 3.71327 | 8.00000 |
| 5.13 | 26.3169 | 2.26495 | 7.16240 | 135.006 | 1.72467 | 3.71569 | 8.00520 |
| 5.14 | 26.4196 | 2.26716 | 7.16938 | 135.797 | 1.72579 | 3.71810 | 8.01040 |
| 5.15 | 26.5225 | 2.26936 | 7.17635 | 136.591 | 1.72691 | 3.72051 | 8.01559 |
| 5.16 | 26.6256 | 2.27156 | 7.18331 | 137.388 | 1.72802 | 3.72292 | 8.02078 |
| 5.17 | 26.7289 | 2.27376 | 7.19027 | 138.188 | 1.72914 | 3.72532 | 8.02596 |
| 5.18 | 26.8324 | 2.27596 | 7.19722 | 138.992 | 1.73025 | 3.72772 | 8.03113 |
| 5.19 | 26.9361 | 2.27816 | 7.20417 | 139.798 | 1.73137 | 3.73012 | 8.03629 |
| **5.20** | 27.0400 | 2.28035 | 7.21110 | 140.608 | 1.73248 | 3.73251 | 8.04145 |
| 5.21 | 27.1441 | 2.28254 | 7.21803 | 141.421 | 1.73359 | 3.73490 | 8.04660 |
| 5.22 | 27.2484 | 2.28473 | 7.22496 | 142.237 | 1.73470 | 3.73729 | 8.05175 |
| 5.23 | 27.3529 | 2.28692 | 7.23187 | 143.056 | 1.73580 | 3.73968 | 8.05689 |
| 5.24 | 27.4576 | 2.28910 | 7.23878 | 143.878 | 1.73691 | 3.74206 | 8.06202 |
| 5.25 | 27.5625 | 2.29129 | 7.24569 | 144.703 | 1.73801 | 3.74443 | 8.06714 |
| 5.26 | 27.6676 | 2.29347 | 7.25259 | 145.532 | 1.73912 | 3.74681 | 8.07226 |
| 5.27 | 27.7729 | 2.29565 | 7.25948 | 146.363 | 1.74022 | 3.74918 | 8.07737 |
| 5.28 | 27.8784 | 2.29783 | 7.26636 | 147.198 | 1.74132 | 3.75155 | 8.08248 |
| 5.29 | 27.9841 | 2.30000 | 7.27324 | 148.036 | 1.74242 | 3.75392 | 8.08758 |
| **5.30** | 28.0900 | 2.30217 | 7.28011 | 148.877 | 1.74351 | 3.75629 | 8.09267 |
| 5.31 | 28.1961 | 2.30434 | 7.28697 | 149.721 | 1.74461 | 3.75865 | 8.09776 |
| 5.32 | 28.3024 | 2.30651 | 7.29383 | 150.569 | 1.74570 | 3.76101 | 8.10284 |
| 5.33 | 28.4089 | 2.30868 | 7.30068 | 151.419 | 1.74680 | 3.76336 | 8.10791 |
| 5.34 | 28.5156 | 2.31084 | 7.30753 | 152.273 | 1.74789 | 3.76571 | 8.11298 |
| 5.35 | 28.6225 | 2.31301 | 7.31437 | 153.130 | 1.74898 | 3.76806 | 8.11804 |
| 5.36 | 28.7296 | 2.31517 | 7.32120 | 153.991 | 1.75007 | 3.77041 | 8.12310 |
| 5.37 | 28.8369 | 2.31733 | 7.32803 | 154.854 | 1.75116 | 3.77275 | 8.12814 |
| 5.38 | 28.9444 | 2.31948 | 7.33485 | 155.721 | 1.75224 | 3.77509 | 8.13319 |
| 5.39 | 29.0521 | 2.32164 | 7.34166 | 156.591 | 1.75333 | 3.77743 | 8.13822 |
| **5.40** | 29.1600 | 2.32379 | 7.34847 | 157.464 | 1.75441 | 3.77976 | 8.14325 |
| 5.41 | 29.2681 | 2.32594 | 7.35527 | 158.340 | 1.75549 | 3.78209 | 8.14828 |
| 5.42 | 29.3764 | 2.32809 | 7.36206 | 159.220 | 1.75657 | 3.78442 | 8.15329 |
| 5.43 | 29.4849 | 2.33024 | 7.36885 | 160.103 | 1.75765 | 3.78675 | 8.15831 |
| 5.44 | 29.5936 | 2.33238 | 7.37564 | 160.989 | 1.75873 | 3.78907 | 8.16331 |
| 5.45 | 29.7025 | 2.33452 | 7.38241 | 161.879 | 1.75981 | 3.79139 | 8.16831 |
| 5.46 | 29.8116 | 2.33666 | 7.38918 | 162.771 | 1.76088 | 3.79371 | 8.17330 |
| 5.47 | 29.9209 | 2.33880 | 7.39594 | 163.667 | 1.76196 | 3.79603 | 8.17829 |
| 5.48 | 30.0304 | 2.34094 | 7.40270 | 164.567 | 1.76303 | 3.79834 | 8.18327 |
| 5.49 | 30.1401 | 2.34307 | 7.40945 | 165.469 | 1.76410 | 3.80065 | 8.18824 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **5.50** | 30.2500 | 2.34521 | 7.41620 | 166.375 | 1.76517 | 3.80295 | 8.19321 |
| 5.51 | 30.3601 | 2.34734 | 7.42294 | 167.284 | 1.76624 | 3.80526 | 8.19818 |
| 5.52 | 30.4704 | 2.34947 | 7.42967 | 168.197 | 1.76731 | 3.80756 | 8.20313 |
| 5.53 | 30.5809 | 2.35160 | 7.43640 | 169.112 | 1.76838 | 3.80985 | 8.20808 |
| 5.54 | 30.6916 | 2.35372 | 7.44312 | 170.031 | 1.76944 | 3.81215 | 8.21303 |
| 5.55 | 30.8025 | 2.35584 | 7.44983 | 170.954 | 1.77051 | 3.81444 | 8.21797 |
| 5.56 | 30.9136 | 2.35797 | 7.45654 | 171.880 | 1.77157 | 3.81673 | 8.22290 |
| 5.57 | 31.0249 | 2.36008 | 7.46324 | 172.809 | 1.77263 | 3.81902 | 8.22783 |
| 5.58 | 31.1364 | 2.36220 | 7.46994 | 173.741 | 1.77369 | 3.82130 | 8.23275 |
| 5.59 | 31.2481 | 2.36432 | 7.47663 | 174.677 | 1.77475 | 3.82358 | 8.23766 |
| **5.60** | 31.3600 | 2.36643 | 7.48331 | 175.616 | 1.77581 | 3.82586 | 8.24257 |
| 5.61 | 31.4721 | 2.36854 | 7.48999 | 176.558 | 1.77686 | 3.82814 | 8.24747 |
| 5.62 | 31.5844 | 2.37065 | 7.49667 | 177.504 | 1.77792 | 3.83041 | 8.25237 |
| 5.63 | 31.6969 | 2.37276 | 7.50333 | 178.454 | 1.77897 | 3.83268 | 8.25726 |
| 5.64 | 31.8096 | 2.37487 | 7.50999 | 179.406 | 1.78003 | 3.83495 | 8.26215 |
| 5.65 | 31.9225 | 2.37697 | 7.51665 | 180.362 | 1.78108 | 3.83722 | 8.26703 |
| 5.66 | 32.0356 | 2.37908 | 7.52330 | 181.321 | 1.78213 | 3.83948 | 8.27190 |
| 5.67 | 32.1489 | 2.38118 | 7.52994 | 182.284 | 1.78318 | 3.84174 | 8.27677 |
| 5.68 | 32.2624 | 2.38328 | 7.53658 | 183.250 | 1.78422 | 3.84399 | 8.28164 |
| 5.69 | 32.3761 | 2.38537 | 7.54321 | 184.220 | 1.78527 | 3.84625 | 8.28649 |
| **5.70** | 32.4900 | 2.38747 | 7.54983 | 185.193 | 1.78632 | 3.84850 | 8.29134 |
| 5.71 | 32.6041 | 2.38956 | 7.55645 | 186.169 | 1.78736 | 3.85075 | 8.29619 |
| 5.72 | 32.7184 | 2.39165 | 7.56307 | 187.149 | 1.78840 | 3.85300 | 8.30103 |
| 5.73 | 32.8329 | 2.39374 | 7.56968 | 188.133 | 1.78944 | 3.85524 | 8.30587 |
| 5.74 | 32.9476 | 2.39583 | 7.57628 | 189.119 | 1.79048 | 3.85748 | 8.31069 |
| 5.75 | 33.0625 | 2.39792 | 7.58288 | 190.109 | 1.79152 | 3.85972 | 8.31552 |
| 5.76 | 33.1776 | 2.40000 | 7.58947 | 191.103 | 1.79256 | 3.86196 | 8.32034 |
| 5.77 | 33.2929 | 2.40208 | 7.59605 | 192.100 | 1.79360 | 3.86419 | 8.32515 |
| 5.78 | 33.4084 | 2.40416 | 7.60263 | 193.101 | 1.79463 | 3.86642 | 8.32995 |
| 5.79 | 33.5241 | 2.40624 | 7.60920 | 194.105 | 1.79567 | 3.86865 | 8.33476 |
| **5.80** | 33.6400 | 2.40832 | 7.61577 | 195.112 | 1.79670 | 3.87088 | 8.33955 |
| 5.81 | 33.7561 | 2.41039 | 7.62234 | 196.123 | 1.79773 | 3.87310 | 8.34434 |
| 5.82 | 33.8724 | 2.41247 | 7.62889 | 197.137 | 1.79876 | 3.87532 | 8.34913 |
| 5.83 | 33.9889 | 2.41454 | 7.63544 | 198.155 | 1.79979 | 3.87754 | 8.35390 |
| 5.84 | 34.1056 | 2 41661 | 7.64199 | 199.177 | 1.80082 | 3.87975 | 8.35868 |
| 5.85 | 34.2225 | 2.41868 | 7.64853 | 200.202 | 1.80185 | 3.88197 | 8.36345 |
| 5.86 | 34.3396 | 2.42074 | 7.65506 | 201.230 | 1.80288 | 3.88418 | 8.36821 |
| 5.87 | 34.4569 | 2.42281 | 7.66159 | 202.262 | 1.80390 | 3.88639 | 8.37297 |
| 5.88 | 34.5744 | 2.42487 | 7.66812 | 203.297 | 1.80492 | 3.88859 | 8.37772 |
| 5.89 | 34.6921 | 2.42693 | 7.67463 | 204.336 | 1.80595 | 3.89080 | 8.38247 |
| **5.90** | 34.8100 | 2.42899 | 7.68115 | 205.379 | 1.80697 | 3.89300 | 8.38721 |
| 5.91 | 34.9281 | 2.43105 | 7.68765 | 206.425 | 1.80799 | 3.89519 | 8.39194 |
| 5.92 | 35.0464 | 2.43311 | 7.69415 | 207.475 | 1.80901 | 3.89739 | 8.39667 |
| 5.93 | 35.1649 | 2.43516 | 7.70065 | 208.528 | 1.81003 | 3.89958 | 8.40140 |
| 5.94 | 35.2836 | 2.43721 | 7.70714 | 209.585 | 1.81104 | 3.90177 | 8.40612 |
| 5.95 | 35.4025 | 2.43926 | 7.71362 | 210.645 | 1.81206 | 3.90396 | 8 41083 |
| 5.96 | 35.5216 | 2.44131 | 7.72010 | 211.709 | 1.81307 | 3.90615 | 8.41554 |
| 5.97 | 35.6409 | 2.44336 | 7.72658 | 212.776 | 1.81409 | 3.90833 | 8.42025 |
| 5.98 | 35.7604 | 2.44540 | 7.73305 | 213.847 | 1.81510 | 3.91051 | 8.42494 |
| 5.99 | 35.8801 | 2.44745 | 7.73951 | 214.922 | 1.81611 | 3.91269 | 8.42964 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **6.00** | 36.0000 | 2.44949 | 7.74597 | 216.000 | 1.81712 | 3.91487 | 8.43433 |
| 6.01 | 36.1201 | 2.45153 | 7.75242 | 217.082 | 1.81813 | 3.91704 | 8.43901 |
| 6.02 | 36.2404 | 2.45357 | 7.75887 | 218.167 | 1.81914 | 3.91921 | 8.44369 |
| 6.03 | 36.3609 | 2.45561 | 7.76531 | 219.256 | 1.82014 | 3.92138 | 8.44836 |
| 6.04 | 36.4816 | 2.45764 | 7.77174 | 220.349 | 1.82115 | 3.92355 | 8.45303 |
| 6.05 | 36.6025 | 2.45967 | 7.77817 | 221.445 | 1.82215 | 3.92571 | 8.45769 |
| 6.06 | 36.7236 | 2.46171 | 7.78460 | 222.545 | 1.82316 | 3.92787 | 8.46235 |
| 6.07 | 36.8449 | 2.46374 | 7.79102 | 223.649 | 1.82416 | 3.93003 | 8.46700 |
| 6.08 | 36.9664 | 2.46577 | 7.79744 | 224.756 | 1.82516 | 3.93219 | 8.47165 |
| 6.09 | 37.0881 | 2.46779 | 7.80385 | 225.867 | 1.82616 | 3.93434 | 8.47629 |
| **6.10** | 37.2100 | 2.46982 | 7.81025 | 226.981 | 1.82716 | 3.93650 | 8.48093 |
| 6.11 | 37.3321 | 2.47184 | 7.81665 | 228.099 | 1.82816 | 3.93865 | 8.48556 |
| 6.12 | 37.4544 | 2.47386 | 7.82304 | 229.221 | 1.82915 | 3.94079 | 8.49018 |
| 6.13 | 37.5769 | 2.47588 | 7.82943 | 230.346 | 1.83015 | 3.94294 | 8.49481 |
| 6.14 | 37.6996 | 2.47790 | 7.83582 | 231.476 | 1.83115 | 3.94508 | 8.49942 |
| 6.15 | 37.8225 | 2.47992 | 7.84219 | 232.608 | 1.83214 | 3.94722 | 8.50403 |
| 6.16 | 37.9456 | 2.48193 | 7.84857 | 233.745 | 1.83313 | 3.94936 | 8.50864 |
| 6.17 | 38.0689 | 2.48395 | 7.85493 | 234.885 | 1.83412 | 3.95150 | 8.51324 |
| 6.18 | 38.1924 | 2.48596 | 7.86130 | 236.029 | 1.83511 | 3.95363 | 8.51784 |
| 6.19 | 38.3161 | 2.48797 | 7.86766 | 237.177 | 1.83610 | 3.95576 | 8.52243 |
| **6.20** | 38.4400 | 2.48998 | 7.87401 | 238.328 | 1.83709 | 3.95789 | 8.52702 |
| 6.21 | 38.5641 | 2.49199 | 7.88036 | 239.483 | 1.83808 | 3.96002 | 8.53160 |
| 6.22 | 38.6884 | 2.49399 | 7.88670 | 240.642 | 1.83906 | 3.96214 | 8.53618 |
| 6.23 | 38.8129 | 2.49600 | 7.89303 | 241.804 | 1.84005 | 3.96427 | 8.54075 |
| 6.24 | 38.9376 | 2.49800 | 7.89937 | 242.971 | 1.84103 | 3.96638 | 8.54532 |
| 6.25 | 39.0625 | 2.50000 | 7.90569 | 244.141 | 1.84202 | 3.96850 | 8.54988 |
| 6.26 | 39.1876 | 2.50200 | 7.91202 | 245.314 | 1.84300 | 3.97062 | 8.55444 |
| 6.27 | 39.3129 | 2.50400 | 7.91833 | 246.492 | 1.84398 | 3.97273 | 8.55899 |
| 6.28 | 39.4384 | 2.50599 | 7.92465 | 247.673 | 1.84496 | 3.97484 | 8.56354 |
| 6.29 | 39.5641 | 2.50799 | 7.93095 | 248.858 | 1.84594 | 3.97695 | 8.56808 |
| **6.30** | 39.6900 | 2.50998 | 7.93725 | 250.047 | 1.84691 | 3.97906 | 8.57262 |
| 6.31 | 39.8161 | 2.51197 | 7.94355 | 251.240 | 1.84789 | 3.98116 | 8.57715 |
| 6.32 | 39.9424 | 2.51396 | 7.94984 | 252.436 | 1.84887 | 3.98326 | 8.58168 |
| 6.33 | 40.0689 | 2.51595 | 7.95613 | 253.636 | 1.84984 | 3.98536 | 8.58620 |
| 6.34 | 40.1956 | 2.51794 | 7.96241 | 254.840 | 1.85082 | 3.98746 | 8.59072 |
| 6.35 | 40.3225 | 2.51992 | 7.96869 | 256.048 | 1.85179 | 3.98956 | 8.59524 |
| 6.36 | 40.4496 | 2.52190 | 7.97496 | 257.259 | 1.85276 | 3.99165 | 8.59975 |
| 6.37 | 40.5769 | 2.52389 | 7.98123 | 258.475 | 1.85373 | 3.99374 | 8.60425 |
| 6.38 | 40.7044 | 2.52587 | 7.98749 | 259.694 | 1.85470 | 3.99583 | 8.60875 |
| 6.39 | 40.8321 | 2.52784 | 7.99375 | 260.917 | 1.85567 | 3.99792 | 8.61325 |
| **6.40** | 40.9600 | 2.52982 | 8.00000 | 262.144 | 1.85664 | 4.00000 | 8.61774 |
| 6.41 | 41.0881 | 2.53180 | 8.00625 | 263.375 | 1.85760 | 4.00208 | 8.62222 |
| 6.42 | 41.2164 | 2.53377 | 8.01249 | 264.609 | 1.85857 | 4.00416 | 8.62671 |
| 6.43 | 41.3449 | 2.53574 | 8.01873 | 265.848 | 1.85953 | 4.00624 | 8.63118 |
| 6.44 | 41.4736 | 2.53772 | 8.02496 | 267.090 | 1.86050 | 4.00832 | 8.63566 |
| 6.45 | 41.6025 | 2.53969 | 8.03119 | 268.336 | 1.86146 | 4.01039 | 8.64012 |
| 6.46 | 41.7316 | 2.54165 | 8.03741 | 269.586 | 1.86242 | 4.01246 | 8.64459 |
| 6.47 | 41.8609 | 2.54362 | 8.04363 | 270.840 | 1.86338 | 4.01453 | 8.64904 |
| 6.48 | 41.9904 | 2.54558 | 8.04984 | 272.098 | 1.86434 | 4.01660 | 8.65350 |
| 6.49 | 42.1201 | 2.54755 | 8.05605 | 273.359 | 1.86530 | 4.01866 | 8.65795 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **6.50** | 42.2500 | 2.54951 | 8.06226 | 274.625 | 1.86626 | 4.02073 | 8.66239 |
| 6.51 | 42.3801 | 2.55147 | 8.06846 | 275.894 | 1.86721 | 4.02279 | 8.66683 |
| 6.52 | 42.5104 | 2.55343 | 8.07465 | 277.168 | 1.86817 | 4.02485 | 8.67127 |
| 6.53 | 42.6409 | 2.55539 | 8.08084 | 278.445 | 1.86912 | 4.02690 | 8.67570 |
| 6.54 | 42.7716 | 2.55734 | 8.08703 | 279.726 | 1.87008 | 4.02896 | 8.68012 |
| 6.55 | 42.9025 | 2.55930 | 8.09321 | 281.011 | 1.87103 | 4.03101 | 8.68455 |
| 6.56 | 43.0336 | 2.56125 | 8.09938 | 282.300 | 1.87198 | 4.03306 | 8.68896 |
| 6.57 | 43.1649 | 2.56320 | 8.10555 | 283.593 | 1.87293 | 4.03511 | 8.69338 |
| 6.58 | 43.2964 | 2.56515 | 8.11172 | 284.890 | 1.87388 | 4.03715 | 8.69778 |
| 6.59 | 43.4281 | 2.56710 | 8.11788 | 286.191 | 1.87483 | 4.03920 | 8.70219 |
| **6.60** | 43.5600 | 2.56905 | 8.12404 | 287.496 | 1.87578 | 4.04124 | 8.70659 |
| 6.61 | 43.6921 | 2.57099 | 8.13019 | 288.805 | 1.87672 | 4.04328 | 8.71098 |
| 6.62 | 43.8244 | 2.57294 | 8.13634 | 290.118 | 1.87767 | 4.04532 | 8.71537 |
| 6.63 | 43.9569 | 2.57488 | 8.14248 | 291.434 | 1.87862 | 4.04735 | 8.71976 |
| 6.64 | 44.0896 | 2.57682 | 8.14862 | 292.755 | 1.87956 | 4.04939 | 8.72414 |
| 6.65 | 44.2225 | 2.57876 | 8.15475 | 294.080 | 1.88050 | 4.05142 | 8.72852 |
| 6.66 | 44.3556 | 2.58070 | 8.16088 | 295.408 | 1.88144 | 4.05345 | 8.73289 |
| 6.67 | 44.4889 | 2.58263 | 8.16701 | 296.741 | 1.88239 | 4.05548 | 8.73726 |
| 6.68 | 44.6224 | 2.58457 | 8.17313 | 298.078 | 1.88333 | 4.05750 | 8.74162 |
| 6.69 | 44.7561 | 2.58650 | 8.17924 | 299.418 | 1.88427 | 4.05953 | 8.74598 |
| **6.70** | 44.8900 | 2.58844 | 8.18535 | 300.763 | 1.88520 | 4.06155 | 8.75034 |
| 6.71 | 45.0241 | 2.59037 | 8.19146 | 302.112 | 1.88614 | 4.06357 | 8.75469 |
| 6.72 | 45.1584 | 2.59230 | 8.19756 | 303.464 | 1.88708 | 4.06559 | 8.75904 |
| 6.73 | 45.2929 | 2.59422 | 8.20366 | 304.821 | 1.88801 | 4.06760 | 8.76338 |
| 6.74 | 45.4276 | 2.59615 | 8.20975 | 306.182 | 1.88895 | 4.06961 | 8.76772 |
| 6.75 | 45.5625 | 2.59808 | 8.21584 | 307.547 | 1.88988 | 4.07163 | 8.77205 |
| 6.76 | 45.6976 | 2.60000 | 8.22192 | 308.916 | 1.89081 | 4.07364 | 8.77638 |
| 6.77 | 45.8329 | 2.60192 | 8.22800 | 310.289 | 1.89175 | 4.07564 | 8.78071 |
| 6.78 | 45.9684 | 2.60384 | 8.23408 | 311.666 | 1.89268 | 4.07765 | 8.78503 |
| 6.79 | 46.1041 | 2.60576 | 8.24015 | 313.047 | 1.89361 | 4.07965 | 8.78935 |
| **6.80** | 46.2400 | 2.60768 | 8.24621 | 314.432 | 1.89454 | 4.08166 | 8.79366 |
| 6.81 | 46.3761 | 2.60960 | 8.25227 | 315.821 | 1.89546 | 4.08365 | 8.79797 |
| 6.82 | 46.5124 | 2.61151 | 8.25833 | 317.215 | 1.89639 | 4.08565 | 8.80227 |
| 6.83 | 46.6489 | 2.61343 | 8.26438 | 318.612 | 1.89732 | 4.08765 | 8.80657 |
| 6.84 | 46.7856 | 2.61534 | 8.27043 | 320.014 | 1.89824 | 4.08964 | 8.81087 |
| 6.85 | 46.9225 | 2.61725 | 8.27647 | 321.419 | 1.89917 | 4.09163 | 8.81516 |
| 6.86 | 47.0596 | 2.61916 | 8.28251 | 322.829 | 1.90009 | 4.09362 | 8.81945 |
| 6.87 | 47.1969 | 2.62107 | 8.28855 | 324.243 | 1.90102 | 4.09561 | 8.82373 |
| 6.88 | 47.3344 | 2.62298 | 8.29458 | 325.661 | 1.90194 | 4.09760 | 8.82801 |
| 6.89 | 47.4721 | 2.62488 | 8.30060 | 327.083 | 1.90286 | 4.09958 | 8.83228 |
| **6.90** | 47.6100 | 2.62679 | 8.30662 | 328.509 | 1.90378 | 4.10157 | 8.83656 |
| 6.91 | 47.7481 | 2.62869 | 8.31264 | 329.939 | 1.90470 | 4.10355 | 8.84082 |
| 6.92 | 47.8864 | 2.63059 | 8.31865 | 331.374 | 1.90562 | 4.10552 | 8.84509 |
| 6.93 | 48.0249 | 2.63249 | 8.32466 | 332.813 | 1.90653 | 4.10750 | 8.84934 |
| 6.94 | 48.1636 | 2.63439 | 8.33067 | 334.255 | 1.90745 | 4.10948 | 8.85360 |
| 6.95 | 48.3025 | 2.63629 | 8.33667 | 335.702 | 1.90837 | 4.11145 | 8.85785 |
| 6.96 | 48.4416 | 2.63818 | 8.34266 | 337.154 | 1.90928 | 4.11342 | 8.86210 |
| 6.97 | 48.5809 | 2.64008 | 8.34865 | 338.609 | 1.91019 | 4.11539 | 8.86634 |
| 6.98 | 48.7204 | 2.64197 | 8.35464 | 340.068 | 1.91111 | 4.11736 | 8.87058 |
| 6.99 | 48.8601 | 2.64386 | 8.36062 | 341.532 | 1.91202 | 4.11932 | 8.87481 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **7.00** | 49.0000 | 2.64575 | 8.36660 | 343.000 | 1.91293 | 4.12129 | 8.87904 |
| 7.01 | 49.1401 | 2.64764 | 8.37257 | 344.472 | 1.91384 | 4.12325 | 8.88327 |
| 7.02 | 49.2804 | 2.64953 | 8.37854 | 345.948 | 1.91475 | 4.12521 | 8.88749 |
| 7.03 | 49.4209 | 2.65141 | 8.38451 | 347.429 | 1.91566 | 4.12716 | 8.89171 |
| 7.04 | 49.5616 | 2.65330 | 8.39047 | 348.914 | 1.91657 | 4.12912 | 8.89592 |
| 7.05 | 49.7025 | 2.65518 | 8.39643 | 350.403 | 1.91747 | 4.13107 | 8.90013 |
| 7.06 | 49.8436 | 2.65707 | 8.40238 | 351.896 | 1.91838 | 4.13303 | 8.90434 |
| 7.07 | 49.9849 | 2.65895 | 8.40833 | 353.393 | 1.91929 | 4.13498 | 8.90854 |
| 7.08 | 50.1264 | 2.66083 | 8.41427 | 354.895 | 1.92019 | 4.13693 | 8.91274 |
| 7.09 | 50.2681 | 2.66271 | 8.42021 | 356.401 | 1.92109 | 4.13887 | 8.91693 |
| **7.10** | 50.4100 | 2.66458 | 8.42615 | 357.911 | 1.92200 | 4.14082 | 8.92112 |
| 7.11 | 50.5521 | 2.66646 | 8.43208 | 359.425 | 1.92290 | 4.14276 | 8.92531 |
| 7.12 | 50.6944 | 2.66833 | 8.43801 | 360.944 | 1.92380 | 4.14470 | 8.92949 |
| 7.13 | 50.8369 | 2.67021 | 8.44393 | 362.467 | 1.92470 | 4.14664 | 8.93367 |
| 7.14 | 50.9796 | 2.67208 | 8.44985 | 363.994 | 1.92560 | 4.14858 | 8.93784 |
| 7.15 | 51.1225 | 2.67395 | 8.45577 | 365.526 | 1.92650 | 4.15052 | 8.94201 |
| 7.16 | 51.2656 | 2.67582 | 8.46168 | 367.062 | 1.92740 | 4.15245 | 8.94618 |
| 7.17 | 51.4089 | 2.67769 | 8.46759 | 368.602 | 1.92829 | 4.15438 | 8.95034 |
| 7.18 | 51.5524 | 2.67955 | 8.47349 | 370.146 | 1.92919 | 4.15631 | 8.95450 |
| 7.19 | 51.6961 | 2.68142 | 8.47939 | 371.695 | 1.93008 | 4.15824 | 8.95866 |
| **7.20** | 51.8400 | 2.68328 | 8.48528 | 373.248 | 1.93098 | 4.16017 | 8.96281 |
| 7.21 | 51.9841 | 2.68514 | 8.49117 | 374.805 | 1.93187 | 4.16209 | 8.96696 |
| 7.22 | 52.1284 | 2.68701 | 8.49706 | 376.367 | 1.93277 | 4.16402 | 8.97110 |
| 7.23 | 52.2729 | 2.68887 | 8.50294 | 377.933 | 1.93366 | 4.16594 | 8.97524 |
| 7.24 | 52.4176 | 2.69072 | 8.50882 | 379.503 | 1.93455 | 4.16786 | 8.97938 |
| 7.25 | 52.5625 | 2.69258 | 8.51469 | 381.078 | 1.93544 | 4.16978 | 8.98351 |
| 7.26 | 52.7076 | 2.69444 | 8.52056 | 382.657 | 1.93633 | 4.17169 | 8.98764 |
| 7.27 | 52.8529 | 2.69629 | 8.52643 | 384.241 | 1.93722 | 4.17361 | 8.99176 |
| 7.28 | 52.9984 | 2.69815 | 8.53229 | 385.828 | 1.93810 | 4.17552 | 8.99588 |
| 7.29 | 53.1441 | 2.70000 | 8.53815 | 387.420 | 1.93899 | 4.17743 | 9.00000 |
| **7.30** | 53.2900 | 2.70185 | 8.54400 | 389.017 | 1.93988 | 4.17934 | 9.00411 |
| 7.31 | 53.4361 | 2.70370 | 8.54985 | 390.618 | 1.94076 | 4.18125 | 9.00822 |
| 7.32 | 53.5824 | 2.70555 | 8.55570 | 392.223 | 1.94165 | 4.18315 | 9.01233 |
| 7.33 | 53.7289 | 2.70740 | 8.56154 | 393.833 | 1.94253 | 4.18506 | 9.01643 |
| 7.34 | 53.8756 | 2.70924 | 8.56738 | 395.447 | 1.94341 | 4.18696 | 9.02053 |
| 7.35 | 54.0225 | 2.71109 | 8.57321 | 397.065 | 1.94430 | 4.18886 | 9.02462 |
| 7.36 | 54.1696 | 2.71293 | 8.57904 | 398.688 | 1.94518 | 4.19076 | 9.02871 |
| 7.37 | 54.3169 | 2.71477 | 8.58487 | 400.316 | 1.94606 | 4.19266 | 9.03280 |
| 7.38 | 54.4644 | 2.71662 | 8.59069 | 401.947 | 1.94694 | 4.19455 | 9.03689 |
| 7.39 | 54.6121 | 2.71846 | 8.59651 | 403.583 | 1.94782 | 4.19644 | 9.04097 |
| **7.40** | 54.7600 | 2.72029 | 8.60233 | 405.224 | 1.94870 | 4.19834 | 9.04504 |
| 7.41 | 54.9081 | 2.72213 | 8.60814 | 406.869 | 1.94957 | 4.20023 | 9.04911 |
| 7.42 | 55.0564 | 2.72397 | 8.61394 | 408.518 | 1.95045 | 4.20212 | 9.05318 |
| 7.43 | 55.2049 | 2.72580 | 8.61974 | 410.172 | 1.95132 | 4.20400 | 9.05725 |
| 7.44 | 55.3536 | 2.72764 | 8.62554 | 411.831 | 1.95220 | 4.20589 | 9.06131 |
| 7.45 | 55.5025 | 2.72947 | 8.63134 | 413.494 | 1.95307 | 4.20777 | 9.06537 |
| 7.46 | 55.6516 | 2.73130 | 8.63713 | 415.161 | 1.95395 | 4.20965 | 9.06942 |
| 7.47 | 55.8009 | 2.73313 | 8.64292 | 416.833 | 1.95482 | 4.21153 | 9.07347 |
| 7.48 | 55.9504 | 2.73496 | 8.64870 | 418.509 | 1.95569 | 4.21341 | 9.07752 |
| 7.49 | 56.1001 | 2.73679 | 8.65448 | 420.190 | 1.95656 | 4.21529 | 9.08156 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **7.50** | 56.2500 | 2.73861 | 8.66025 | 421.875 | 1.95743 | 4.21716 | 9.08560 |
| 7.51 | 56.4001 | 2.74044 | 8.66603 | 423.565 | 1.95830 | 4.21904 | 9.08964 |
| 7.52 | 56.5504 | 2.74226 | 8.67179 | 425.259 | 1.95917 | 4 22091 | 9.09367 |
| 7.53 | 56.7009 | 2.74408 | 8.67756 | 426.958 | 1.96004 | 4.22278 | 9.09770 |
| 7.54 | 56.8516 | 2.74591 | 8.68332 | 428.661 | 1.96091 | 4.22465 | 9.10173 |
| 7.55 | 57.0025 | 2.74773 | 8.68907 | 430.369 | 1.96177 | 4.22651 | 9.10575 |
| 7.56 | 57.1536 | 2.74955 | 8.69483 | 432.081 | 1.96264 | 4.22838 | 9.10977 |
| 7.57 | 57.3049 | 2.75136 | 8.70057 | 433.798 | 1.96350 | 4.23024 | 9.11378 |
| 7.58 | 57.4564 | 2.75318 | 8.70632 | 435.520 | 1.96437 | 4.23210 | 9.11779 |
| 7.59 | 57.6081 | 2.75500 | 8.71206 | 437.245 | 1.96523 | 4.23396 | 9.12180 |
| **7.60** | 57.7600 | 2.75681 | 8.71780 | 438.976 | 1.96610 | 4.23582 | 9.12581 |
| 7.61 | 57.9121 | 2.75862 | 8.72353 | 440.711 | 1.96696 | 4.23768 | 9.12981 |
| 7.62 | 58.0644 | 2.76043 | 8.72926 | 442.451 | 1.96782 | 4.23954 | 9.13380 |
| 7.63 | 58.2169 | 2.76225 | 8.73499 | 444.195 | 1.96868 | 4.24139 | 9.13780 |
| 7.64 | 58.3696 | 2.76405 | 8.74071 | 445.944 | 1.96954 | 4.24324 | 9.14179 |
| 7.65 | 58.5225 | 2.76586 | 8.74643 | 447.697 | 1.97040 | 4.24509 | 9.14577 |
| 7.66 | 58.6756 | 2.76767 | 8.75214 | 449.455 | 1.97126 | 4.24694 | 9.14976 |
| 7.67 | 58.8289 | 2.76948 | 8.75785 | 451.218 | 1.97211 | 4.24879 | 9.15374 |
| 7.68 | 58.9824 | 2.77128 | 8.76356 | 452.985 | 1.97297 | 4.25063 | 9.15771 |
| 7.69 | 59.1361 | 2.77308 | 8.76926 | 454.757 | 1.97383 | 4.25248 | 9.16169 |
| **7.70** | 59.2900 | 2.77489 | 8.77496 | 456.533 | 1.97468 | 4.25432 | 9.16566 |
| 7.71 | 59.4441 | 2.77669 | 8.78066 | 458.314 | 1.97554 | 4.25616 | 9.16962 |
| 7.72 | 59.5984 | 2.77849 | 8.78635 | 460.100 | 1.97639 | 4.25800 | 9.17359 |
| 7.73 | 59.7529 | 2.78029 | 8.79204 | 461.890 | 1.97724 | 4.25984 | 9.17754 |
| 7.74 | 59.9076 | 2.78209 | 8.79773 | 463.685 | 1.97809 | 4.26167 | 9.18150 |
| 7.75 | 60.0625 | 2.78388 | 8.80341 | 465.484 | 1.97895 | 4.26351 | 9.18545 |
| 7.76 | 60.2176 | 2.78568 | 8.80909 | 467.289 | 1.97980 | 4.26534 | 9.18940 |
| 7.77 | 60.3729 | 2.78747 | 8.81476 | 469.097 | 1.98065 | 4.26717 | 9.19335 |
| 7.78 | 60.5284 | 2.78927 | 8.82043 | 470.911 | 1.98150 | 4.26900 | 9.19729 |
| 7.79 | 60.6841 | 2.79106 | 8.82610 | 472.729 | 1.98234 | 4.27083 | 9.20123 |
| **7.80** | 60.8400 | 2.79285 | 8.83176 | 474.552 | 1.98319 | 4.27266 | 9.20516 |
| 7.81 | 60.9961 | 2.79464 | 8.83742 | 476.380 | 1.98404 | 4.27448 | 9.20910 |
| 7.82 | 61.1524 | 2.79643 | 8.84308 | 478.212 | 1.98489 | 4.27631 | 9.21302 |
| 7.83 | 61.3089 | 2.79821 | 8.84873 | 480.049 | 1.98573 | 4.27813 | 9.21695 |
| 7.84 | 61.4656 | 2.80000 | 8.85438 | 481.890 | 1.98658 | 4.27995 | 9.22087 |
| 7.85 | 61.6225 | 2.80179 | 8.86002 | 483.737 | 1.98742 | 4.28177 | 9.22479 |
| 7.86 | 61.7796 | 2.80357 | 8.86566 | 485.588 | 1.98826 | 4.28359 | 9.22871 |
| 7.87 | 61.9369 | 2.80535 | 8.87130 | 487.443 | 1.98911 | 4.28540 | 9.23262 |
| 7.88 | 62.0944 | 2.80713 | 8.87694 | 489.304 | 1.98995 | 4.28722 | 9.23653 |
| 7.89 | 62.2521 | 2.80891 | 8.88257 | 491.169 | 1.99079 | 4.28903 | 9.24043 |
| **7.90** | 62.4100 | 2.81069 | 8.88819 | 493.039 | 1.99163 | 4.29084 | 9.24434 |
| 7.91 | 62.5681 | 2.81247 | 8.89382 | 494.914 | 1.99247 | 4.29265 | 9.24823 |
| 7.92 | 62.7264 | 2.81425 | 8.89944 | 496.793 | 1.99331 | 4.29446 | 9.25213 |
| 7.93 | 62.8849 | 2.81603 | 8.90505 | 498.677 | 1.99415 | 4.29627 | 9.25602 |
| 7.94 | 63.0436 | 2.81780 | 8.91067 | 500.566 | 1.99499 | 4.29807 | 9.25991 |
| 7.95 | 63.2025 | 2.81957 | 8.91628 | 502.460 | 1.99582 | 4.29987 | 9.26380 |
| 7.96 | 63.3616 | 2.82135 | 8.92188 | 504.358 | 1.99666 | 4.30168 | 9.26768 |
| 7.97 | 63.5209 | 2.82312 | 8.92749 | 506.262 | 1.99750 | 4.30348 | 9.27156 |
| 7.98 | 63.6804 | 2.82489 | 8.93308 | 508.170 | 1.99833 | 4.30528 | 9.27544 |
| 7.99 | 63.8401 | 2.82666 | 8.93868 | 510.082 | 1.99917 | 4.30707 | 9.27931 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **8.00** | 64.0000 | 2.82843 | 8.94427 | 512.000 | 2.00000 | 4.30887 | 9.28318 |
| 8.01 | 64.1601 | 2.83019 | 8.94986 | 513.922 | 2.00083 | 4.31066 | 9.28704 |
| 8.02 | 64.3204 | 2.83196 | 8.95545 | 515.850 | 2.00167 | 4.31246 | 9.29091 |
| 8.03 | 64.4809 | 2.83373 | 8.96103 | 517.782 | 2.00250 | 4.31425 | 9.29477 |
| 8.04 | 64.6416 | 2.83549 | 8.96660 | 519.718 | 2.00333 | 4.31604 | 9.29862 |
| 8.05 | 64.8025 | 2.83725 | 8.97218 | 521.660 | 2.00416 | 4.31783 | 9.30248 |
| 8.06 | 64.9636 | 2.83901 | 8.97775 | 523.607 | 2.00499 | 4.31961 | 9.30633 |
| 8.07 | 65.1249 | 2.84077 | 8.98332 | 525.558 | 2.00582 | 4.32140 | 9.31018 |
| 8.08 | 65.2864 | 2.84253 | 8.98888 | 527.514 | 2.00664 | 4.32318 | 9.31402 |
| 8.09 | 65.4481 | 2.84429 | 8.99444 | 529.475 | 2.00747 | 4.32497 | 9.31786 |
| **8.10** | 65.6100 | 2.84605 | 9.00000 | 531.441 | 2.00830 | 4.32675 | 9.32170 |
| 8.11 | 65.7721 | 2.84781 | 9.00555 | 533.412 | 2.00912 | 4.32853 | 9.32553 |
| 8.12 | 65.9344 | 2.84956 | 9.01110 | 535.387 | 2.00995 | 4.33031 | 9.32936 |
| 8.13 | 66.0969 | 2.85132 | 9.01665 | 537.368 | 2.01078 | 4.33208 | 9.33319 |
| 8.14 | 66.2596 | 2.85307 | 9.02219 | 539.353 | 2.01160 | 4.33386 | 9.33702 |
| 8.15 | 66.4225 | 2.85482 | 9.02774 | 541.343 | 2.01242 | 4.33563 | 9.34084 |
| 8.16 | 66.5856 | 2.85657 | 9.03327 | 543.338 | 2.01325 | 4.33741 | 9.34466 |
| 8.17 | 66.7489 | 2.85832 | 9.03881 | 545.339 | 2.01407 | 4.33918 | 9.34847 |
| 8.18 | 66.9124 | 2.86007 | 9.04434 | 547.343 | 2.01489 | 4.34095 | 9.35229 |
| 8.19 | 67.0761 | 2.86182 | 9.04986 | 549.353 | 2.01571 | 4.34271 | 9.35610 |
| **8.20** | 67.2400 | 2.86356 | 9.05539 | 551.368 | 2.01653 | 4.34448 | 9.35990 |
| 8.21 | 67.4041 | 2.86531 | 9.06091 | 553.388 | 2.01735 | 4.34625 | 9.36370 |
| 8.22 | 67.5684 | 2.86705 | 9.06642 | 555.412 | 2.01817 | 4.34801 | 9.36751 |
| 8.23 | 67.7329 | 2.86880 | 9.07193 | 557.442 | 2.01899 | 4.34977 | 9.37130 |
| 8.24 | 67.8976 | 2.87054 | 9.07744 | 559.476 | 2.01980 | 4.35153 | 9.37510 |
| 8.25 | 68.0625 | 2.87228 | 9.08295 | 561.516 | 2.02062 | 4.35329 | 9.37889 |
| 8.26 | 68.2276 | 2.87402 | 9.08845 | 563.560 | 2.02144 | 4.35505 | 9.38268 |
| 8.27 | 68.3929 | 2.87576 | 9.09395 | 565.609 | 2.02225 | 4.35681 | 9.38646 |
| 8.28 | 68.5584 | 2.87750 | 9.09945 | 567.664 | 2.02307 | 4.35856 | 9.39024 |
| 8.29 | 68.7241 | 2.87924 | 9.10494 | 569.723 | 2.02388 | 4.36032 | 9.39402 |
| **8.30** | 68.8900 | 2.88097 | 9.11043 | 571.787 | 2.02469 | 4.36207 | 9.39780 |
| 8.31 | 69.0561 | 2.88271 | 9.11592 | 573.856 | 2.02551 | 4.36382 | 9.40157 |
| 8.32 | 69.2224 | 2.88444 | 9.12140 | 575.930 | 2.02632 | 4.36557 | 9.40534 |
| 8.33 | 69.3889 | 2.88617 | 9.12688 | 578.010 | 2.02713 | 4.36732 | 9.40911 |
| 8.34 | 69.5556 | 2.88791 | 9.13236 | 580.094 | 2.02794 | 4.36907 | 9.41287 |
| 8.35 | 69.7225 | 2.88964 | 9.13783 | 582.183 | 2.02875 | 4.37081 | 9.41663 |
| 8.36 | 69.8896 | 2.89137 | 9.14330 | 584.277 | 2.02956 | 4.37256 | 9.42039 |
| 8.37 | 70.0569 | 2.89310 | 9.14877 | 586.376 | 2.03037 | 4.37430 | 9.42414 |
| 8.38 | 70.2244 | 2.89482 | 9.15423 | 588.480 | 2.03118 | 4.37604 | 9.42789 |
| 8.39 | 70.3921 | 2.89655 | 9.15969 | 590.590 | 2.03199 | 4.37778 | 9.43164 |
| **8.40** | 70.5600 | 2.89828 | 9.16515 | 592.704 | 2.03279 | 4.37952 | 9.43539 |
| 8.41 | 70.7281 | 2.90000 | 9.17061 | 594.823 | 2.03360 | 4.38126 | 9.43913 |
| 8.42 | 70.8964 | 2.90172 | 9.17606 | 596.948 | 2.03440 | 4.38299 | 9.44287 |
| 8.43 | 71.0649 | 2.90345 | 9.18150 | 599.077 | 2.03521 | 4.38473 | 9.44661 |
| 8.44 | 71.2336 | 2.90517 | 9.18695 | 601.212 | 2.03601 | 4.38646 | 9.45034 |
| 8.45 | 71.4025 | 2.90689 | 9.19239 | 603.351 | 2.03682 | 4.38819 | 9.45407 |
| 8.46 | 71.5716 | 2.90861 | 9.19783 | 605.496 | 2.03762 | 4.38992 | 9.45780 |
| 8.47 | 71.7409 | 2.91033 | 9.20326 | 607.645 | 2.03842 | 4.39165 | 9.46152 |
| 8.48 | 71.9104 | 2.91204 | 9.20869 | 609.800 | 2.03923 | 4.39338 | 9.46525 |
| 8.49 | 72.0801 | 2.91376 | 9.21412 | 611.960 | 2.04003 | 4.39510 | 9.46897 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **8.50** | 72.2500 | 2.91548 | 9.21954 | 614.125 | 2.04083 | 4.39683 | 9.47268 |
| 8.51 | 72.4201 | 2.91719 | 9.22497 | 616.295 | 2.04163 | 4.39855 | 9.47640 |
| 8.52 | 72.5904 | 2.91890 | 9.23038 | 618.470 | 2.04243 | 4.40028 | 9.48011 |
| 8.53 | 72.7609 | 2.92062 | 9.23580 | 620.650 | 2.04323 | 4.40200 | 9.48381 |
| 8.54 | 72.9316 | 2.92233 | 9.24121 | 622.836 | 2.04402 | 4.40372 | 9.48752 |
| 8.55 | 73.1025 | 2.92404 | 9.24662 | 625.026 | 2.04482 | 4.40543 | 9.49122 |
| 8.56 | 73.2736 | 2.92575 | 9.25203 | 627.222 | 2.04562 | 4.40715 | 9.49492 |
| 8.57 | 73.4449 | 2.92746 | 9.25743 | 629.423 | 2.04641 | 4.40887 | 9.49861 |
| 8.58 | 73.6164 | 2.92916 | 9.26283 | 631.629 | 2.04721 | 4.41058 | 9.50231 |
| 8.59 | 73.7881 | 2.93087 | 9.26823 | 633.840 | 2.04801 | 4.41229 | 9.50600 |
| **8.60** | 73.9600 | 2.93258 | 9.27362 | 636.056 | 2.04880 | 4.41400 | 9.50969 |
| 8.61 | 74.1321 | 2.93428 | 9.27901 | 638.277 | 2.04959 | 4.41571 | 9.51337 |
| 8.62 | 74.3044 | 2.93598 | 9.28440 | 640.504 | 2.05039 | 4.41742 | 9.51705 |
| 8.63 | 74.4769 | 2.93769 | 9.28978 | 642.736 | 2.05118 | 4.41913 | 9.52073 |
| 8.64 | 74.6496 | 2.93939 | 9.29516 | 644.973 | 2.05197 | 4.42084 | 9.52441 |
| 8.65 | 74.8225 | 2.94109 | 9.30054 | 647.215 | 2.05276 | 4.42254 | 9.52808 |
| 8.66 | 74.9956 | 2.94279 | 9.30591 | 649.462 | 2.05355 | 4.42425 | 9.53175 |
| 8.67 | 75.1689 | 2.94449 | 9.31128 | 651.714 | 2.05434 | 4.42595 | 9.53542 |
| 8.68 | 75.3424 | 2.94618 | 9.31665 | 653.972 | 2.05513 | 4.42765 | 9.53908 |
| 8.69 | 75.5161 | 2.94788 | 9.32202 | 656.235 | 2.05592 | 4.42935 | 9.54274 |
| **8.70** | 75.6900 | 2.94958 | 9.32738 | 658.503 | 2.05671 | 4.43105 | 9.54640 |
| 8.71 | 75.8641 | 2.95127 | 9.33274 | 660.776 | 2.05750 | 4.43274 | 9.55006 |
| 8.72 | 76.0384 | 2.95296 | 9.33809 | 663.055 | 2.05828 | 4.43444 | 9.55371 |
| 8.73 | 76.2129 | 2.95466 | 9.34345 | 665.339 | 2.05907 | 4.43613 | 9.55736 |
| 8.74 | 76.3876 | 2.95635 | 9.34880 | 667.628 | 2.05986 | 4.43783 | 9.56101 |
| 8.75 | 76.5625 | 2.95804 | 9.35414 | 669.922 | 2.06064 | 4.43952 | 9.56466 |
| 8.76 | 76.7376 | 2.95973 | 9.35949 | 672.221 | 2.06143 | 4.44121 | 9.56830 |
| 8.77 | 76.9129 | 2.96142 | 9.36483 | 674.526 | 2.06221 | 4.44290 | 9.57194 |
| 8.78 | 77.0884 | 2.96311 | 9.37017 | 676.836 | 2.06299 | 4.44459 | 9.57557 |
| 8.79 | 77.2641 | 2.96479 | 9.37550 | 679.151 | 2.06378 | 4.44627 | 9.57921 |
| **8.80** | 77.4400 | 2.96648 | 9.38083 | 681.472 | 2.06456 | 4.44796 | 9.58284 |
| 8.81 | 77.6161 | 2.96816 | 9.38616 | 683.798 | 2.06534 | 4.44964 | 9.58647 |
| 8.82 | 77.7924 | 2.96985 | 9.39149 | 686.129 | 2.06612 | 4.45133 | 9.59009 |
| 8.83 | 77.9689 | 2.97153 | 9.39681 | 688.465 | 2.06690 | 4.45301 | 9.59372 |
| 8.84 | 78.1456 | 2.97321 | 9.40213 | 690.807 | 2.06768 | 4.45469 | 9.59734 |
| 8.85 | 78.3225 | 2.97489 | 9.40744 | 693.154 | 2.06846 | 4.45637 | 9.60095 |
| 8.86 | 78.4996 | 2.97658 | 9.41276 | 695.506 | 2.06924 | 4.45805 | 9.60457 |
| 8.87 | 78.6769 | 2.97825 | 9.41807 | 697.864 | 2.07002 | 4.45972 | 9.60818 |
| 8.88 | 78.8544 | 2.97993 | 9.42338 | 700.227 | 2.07080 | 4.46140 | 9.61179 |
| 8.89 | 79.0321 | 2.98161 | 9.42868 | 702.595 | 2.07157 | 4.46307 | 9.61540 |
| **8.90** | 79.2100 | 2.98329 | 9.43398 | 704.969 | 2.07235 | 4.46475 | 9.61900 |
| 8.91 | 79.3881 | 2.98496 | 9.43928 | 707.348 | 2.07313 | 4.46642 | 9.62260 |
| 8.92 | 79.5664 | 2.98664 | 9.44458 | 709.732 | 2.07390 | 4.46809 | 9.62620 |
| 8.93 | 79.7449 | 2.98831 | 9.44987 | 712.122 | 2.07468 | 4.46976 | 9.62980 |
| 8.94 | 79.9236 | 2.98998 | 9.45516 | 714.517 | 2.07545 | 4.47142 | 9.63339 |
| 8.95 | 80.1025 | 2.99166 | 9.46044 | 716.917 | 2.07622 | 4.47309 | 9.63698 |
| 8.96 | 80.2816 | 2.99333 | 9.46573 | 719.323 | 2.07700 | 4.47476 | 9.64057 |
| 8.97 | 80.4609 | 2.99500 | 9.47101 | 721.734 | 2.07777 | 4.47642 | 9.64415 |
| 8.98 | 80.6404 | 2.99666 | 9.47629 | 724.151 | 2.07854 | 4.47808 | 9.64774 |
| 8.99 | 80.8201 | 2.99833 | 9.48156 | 726.573 | 2.07931 | 4.47974 | 9.65132 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **9.00** | 81.0000 | 3.00000 | 9.48683 | 729.000 | 2.08008 | 4.48140 | 9.65489 |
| 9.01 | 81.1801 | 3.00167 | 9.49210 | 731.433 | 2.08085 | 4.48306 | 9.65847 |
| 9.02 | 81.3604 | 3.00333 | 9.49737 | 733.871 | 2.08162 | 4.48472 | 9.66204 |
| 9.03 | 81.5409 | 3.00500 | 9.50263 | 736.314 | 2.08239 | 4.48638 | 9.66561 |
| 9.04 | 81.7216 | 3.00666 | 9.50789 | 738.763 | 2.08316 | 4.48803 | 9.66918 |
| 9.05 | 81.9025 | 3.00832 | 9.51315 | 741.218 | 2.08393 | 4.48969 | 9.67274 |
| 9.06 | 82.0836 | 3.00998 | 9.51840 | 743.677 | 2.08470 | 4.49134 | 9.67630 |
| 9.07 | 82.2649 | 3.01164 | 9.52365 | 746.143 | 2.08546 | 4.49299 | 9.67986 |
| 9.08 | 82.4464 | 3.01330 | 9.52890 | 748.613 | 2.08623 | 4.49464 | 9.68342 |
| 9.09 | 82.6281 | 3.01496 | 9.53415 | 751.089 | 2.08699 | 4.49629 | 9.68697 |
| **9.10** | 82.8100 | 3.01662 | 9.53939 | 753.571 | 2.08776 | 4.49794 | 9.69052 |
| 9.11 | 82.9921 | 3.01828 | 9.54463 | 756.058 | 2.08852 | 4.49959 | 9.69407 |
| 9.12 | 83.1744 | 3.01993 | 9.54987 | 758.551 | 2.08929 | 4.50123 | 9.69762 |
| 9.13 | 83.3569 | 3.02159 | 9.55510 | 761.048 | 2.09005 | 4.50288 | 9.70116 |
| 9.14 | 83.5396 | 3.02324 | 9.56033 | 763.552 | 2.09081 | 4.50452 | 9.70470 |
| 9.15 | 83.7225 | 3.02490 | 9.56556 | 766.061 | 2.09158 | 4.50616 | 9.70824 |
| 9.16 | 83.9056 | 3.02655 | 9.57079 | 768.575 | 2.09234 | 4.50781 | 9.71177 |
| 9.17 | 84.0889 | 3.02820 | 9.57601 | 771.095 | 2.09310 | 4.50945 | 9.71531 |
| 9.18 | 84.2724 | 3.02985 | 9.58123 | 773.621 | 2.09386 | 4.51108 | 9.71884 |
| 9.19 | 84.4561 | 3.03150 | 9.58645 | 776.152 | 2.09462 | 4.51272 | 9.72236 |
| **9.20** | 84.6400 | 3.03315 | 9.59166 | 778.688 | 2.09538 | 4.51436 | 9.72589 |
| 9.21 | 84.8241 | 3.03480 | 9.59687 | 781.230 | 2.09614 | 4.51599 | 9.72941 |
| 9.22 | 85.0084 | 3.03645 | 9.60208 | 783.777 | 2.09690 | 4.51763 | 9.73293 |
| 9.23 | 85.1929 | 3.03809 | 9.60729 | 786.330 | 2.09765 | 4.51926 | 9.73645 |
| 9.24 | 85.3776 | 3.03974 | 9.61249 | 788.889 | 2.09841 | 4.52089 | 9.73996 |
| 9.25 | 85.5625 | 3.04138 | 9.61769 | 791.453 | 2.09917 | 4.52252 | 9.74348 |
| 9.26 | 85.7476 | 3.04302 | 9.62289 | 794.023 | 2.09992 | 4.52415 | 9.74699 |
| 9.27 | 85.9329 | 3.04467 | 9.62808 | 796.598 | 2.10068 | 4.52578 | 9.75049 |
| 9.28 | 86.1184 | 3.04631 | 9.63328 | 799.179 | 2.10144 | 4.52740 | 9.75400 |
| 9.29 | 86.3041 | 3.04795 | 9.63846 | 801.765 | 2.10219 | 4.52903 | 9.75750 |
| **9.30** | 86.4900 | 3.04959 | 9.64365 | 804.357 | 2.10294 | 4.53065 | 9.76100 |
| 9.31 | 86.6761 | 3.05123 | 9.64883 | 806.954 | 2.10370 | 4.53228 | 9.764[illegible] |
| 9.32 | 86.8624 | 3.05287 | 9.65401 | 809.558 | 2.10445 | 4.53390 | 9.7679[illegible] |
| 9.33 | 87.0489 | 3.05450 | 9.65919 | 812.166 | 2.10520 | 4.53552 | 9.77148 |
| 9.34 | 87.2356 | 3.05614 | 9.66437 | 814.781 | 2.10595 | 4.53714 | 9.77497 |
| 9.35 | 87.4225 | 3.05778 | 9.66954 | 817.400 | 2.10671 | 4.53876 | 9.77846 |
| 9.36 | 87.6096 | 3.05941 | 9.67471 | 820.026 | 2.10746 | 4.54038 | 9.78195 |
| 9.37 | 87.7969 | 3.06105 | 9.67988 | 822.657 | 2.10821 | 4.54199 | 9.78543 |
| 9.38 | 87.9844 | 3.06268 | 9.68504 | 825.294 | 2.10896 | 4.54361 | 9.78891 |
| 9.39 | 88.1721 | 3.06431 | 9.69020 | 827.936 | 2.10971 | 4.54522 | 9.79239 |
| **9.40** | 88.3600 | 3.06594 | 9.69536 | 830.584 | 2.11045 | 4.54684 | 9.79586 |
| 9.41 | 88.5481 | 3.06757 | 9.70052 | 833.238 | 2.11120 | 4.54845 | 9.79933 |
| 9.42 | 88.7364 | 3.06920 | 9.70567 | 835.897 | 2.11195 | 4.55006 | 9.80280 |
| 9.43 | 88.9249 | 3.07083 | 9.71082 | 838.562 | 2.11270 | 4.55167 | 9.80627 |
| 9.44 | 89.1136 | 3.07246 | 9.71597 | 841.232 | 2.11344 | 4.55328 | 9.80974 |
| 9.45 | 89.3025 | 3.07409 | 9.72111 | 843.909 | 2.11419 | 4.55488 | 9.81320 |
| 9.46 | 89.4916 | 3.07571 | 9.72625 | 846.591 | 2.11494 | 4.55649 | 9.81666 |
| 9.47 | 89.6809 | 3.07734 | 9.73139 | 849.278 | 2.11568 | 4.55809 | 9.82012 |
| 9.48 | 89.8704 | 3.07896 | 9.73653 | 851.971 | 2.11642 | 4.55970 | 9.82357 |
| 9.49 | 90.0601 | 3.08058 | 9.74166 | 854.670 | 2.11717 | 4.56130 | 9.82703 |

| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n^3$ | $\sqrt[3]{n}$ | $\sqrt[3]{10n}$ | $\sqrt[3]{100n}$ |
|---|---|---|---|---|---|---|---|
| **9.50** | 90.2500 | 3.08221 | 9.74679 | 857.375 | 2.11791 | 4.56290 | 9.83048 |
| 9.51 | 90.4401 | 3.08383 | 9.75192 | 860.085 | 2.11865 | 4.56450 | 9.83392 |
| 9.52 | 90.6304 | 3.08545 | 9.75705 | 862.801 | 2.11940 | 4.56610 | 9.83737 |
| 9.53 | 90.8209 | 3.08707 | 9.76217 | 865.523 | 2.12014 | 4.56770 | 9.84081 |
| 9.54 | 91.0116 | 3.08869 | 9.76729 | 868.251 | 2.12088 | 4.56930 | 9.84425 |
| 9.55 | 91.2025 | 3.09031 | 9.77241 | 870.984 | 2.12162 | 4.57089 | 9.84769 |
| 9.56 | 91.3936 | 3.09192 | 9.77753 | 873.723 | 2.12236 | 4.57249 | 9.85113 |
| 9.57 | 91.5849 | 3.09354 | 9.78264 | 876.467 | 2.12310 | 4.57408 | 9.85456 |
| 9.58 | 91.7764 | 3.09516 | 9.78775 | 879.218 | 2.12384 | 4.57567 | 9.85799 |
| 9.59 | 91.9681 | 3.09677 | 9.79285 | 881.974 | 2.12458 | 4.57727 | 9.86142 |
| **9.60** | 92.1600 | 3.09839 | 9.79796 | 884.736 | 2.12532 | 4.57886 | 9.86485 |
| 9.61 | 92.3521 | 3.10000 | 9.80306 | 887.504 | 2.12605 | 4.58045 | 9.86827 |
| 9.62 | 92.5444 | 3.10161 | 9.80816 | 890.277 | 2.12679 | 4.58204 | 9.87169 |
| 9.63 | 92.7369 | 3.10322 | 9.81326 | 893.056 | 2.12753 | 4.58362 | 9.87511 |
| 9.64 | 92.9296 | 3.10483 | 9.81835 | 895.841 | 2.12826 | 4.58521 | 9.87853 |
| 9.65 | 93.1225 | 3.10644 | 9.82344 | 898.632 | 2.12900 | 4.58679 | 9.88195 |
| 9.66 | 93.3156 | 3.10805 | 9.82853 | 901.429 | 2.12974 | 4.58838 | 9.88536 |
| 9.67 | 93.5089 | 3.10966 | 9.83362 | 904.231 | 2.13047 | 4.58996 | 9.88877 |
| 9.68 | 93.7024 | 3.11127 | 9.83870 | 907.039 | 2.13120 | 4.59154 | 9.89217 |
| 9.69 | 93.8961 | 3.11288 | 9.84378 | 909.853 | 2.13194 | 4.59312 | 9.89558 |
| **9.70** | 94.0900 | 3.11448 | 9.84886 | 912.673 | 2.13267 | 4.59470 | 9.89898 |
| 9.71 | 94.2841 | 3.11609 | 9.85393 | 915.499 | 2.13340 | 4.59628 | 9.90238 |
| 9.72 | 94.4784 | 3.11769 | 9.85901 | 918.330 | 2.13414 | 4.59786 | 9.90578 |
| 9.73 | 94.6729 | 3.11929 | 9.86408 | 921.167 | 2.13487 | 4.59943 | 9.90918 |
| 9.74 | 94.8676 | 3.12090 | 9.86914 | 924.010 | 2.13560 | 4.60101 | 9.91257 |
| 9.75 | 95.0625 | 3.12250 | 9.87421 | 926.859 | 2.13633 | 4.60258 | 9.91596 |
| 9.76 | 95.2576 | 3.12410 | 9.87927 | 929.714 | 2.13706 | 4.60416 | 9.91935 |
| 9.77 | 95.4529 | 3.12570 | 9.88433 | 932.575 | 2.13779 | 4.60573 | 9.92274 |
| 9.78 | 95.6484 | 3.12730 | 9.88939 | 935.441 | 2.13852 | 4.60730 | 9.92612 |
| 9.79 | 95.8441 | 3.12890 | 9.89444 | 938.314 | 2.13925 | 4.60887 | 9.92950 |
| **9.80** | 96.0400 | 3.13050 | 9.89949 | 941.192 | 2.13997 | 4.61044 | 9.93288 |
| 9.81 | 96.2361 | 3.13209 | 9.90454 | 944.076 | 2.14070 | 4.61200 | 9.93626 |
| 9.82 | 96.4324 | 3.13369 | 9.90959 | 946.966 | 2.14143 | 4.61357 | 9.93964 |
| 9.83 | 96.6289 | 3.13528 | 9.91464 | 949.862 | 2.14216 | 4.61514 | 9.94301 |
| 9.84 | 96.8256 | 3.13688 | 9.91968 | 952.764 | 2.14288 | 4.61670 | 9.94638 |
| 9.85 | 97.0225 | 3.13847 | 9.92472 | 955.672 | 2.14361 | 4.61826 | 9.94975 |
| 9.86 | 97.2196 | 3.14006 | 9.92975 | 958.585 | 2.14433 | 4.61983 | 9.95311 |
| 9.87 | 97.4169 | 3.14166 | 9.93479 | 961.505 | 2.14506 | 4.62139 | 9.95648 |
| 9.88 | 97.6144 | 3.14325 | 9.93982 | 964.430 | 2.14578 | 4.62295 | 9.95984 |
| 9.89 | 97.8121 | 3.14484 | 9.94485 | 967.362 | 2.14651 | 4.62451 | 9.96320 |
| **9.90** | 98.0100 | 3.14643 | 9.94987 | 970.299 | 2.14723 | 4.62607 | 9.96655 |
| 9.91 | 98.2081 | 3.14802 | 9.95490 | 973.242 | 2.14795 | 4.62762 | 9.96991 |
| 9.92 | 98.4064 | 3.14960 | 9.95992 | 976.191 | 2.14867 | 4.62918 | 9.97326 |
| 9.93 | 98.6049 | 3.15119 | 9.96494 | 979.147 | 2.14940 | 4.63073 | 9.97661 |
| 9.94 | 98.8036 | 3.15278 | 9.96995 | 982.108 | 2.15012 | 4.63229 | 9.97996 |
| 9.95 | 99.0025 | 3.15436 | 9.97497 | 985.075 | 2.15084 | 4.63384 | 9.98331 |
| 9.96 | 99.2016 | 3.15595 | 9.97998 | 988.048 | 2.15156 | 4.63539 | 9.98665 |
| 9.97 | 99.4009 | 3.15753 | 9.98499 | 991.027 | 2.15228 | 4.63694 | 9.98999 |
| 9.98 | 99.6004 | 3.15911 | 9.98999 | 994.012 | 2.15300 | 4.63849 | 9.99333 |
| 9.99 | 99.8001 | 3.16070 | 9.99500 | 997.003 | 2.15372 | 4.64004 | 9.99667 |

# TABLE II — IMPORTANT NUMBERS

## *A. Units of Length*

### English Units

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards = 1 rod (rd.)
320 rods = 1 mile (mi.)

### Metric Units

10 millimeters (mm.) = 1 centimeter (cm.)
10 centimeters = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
10 meters = 1 dekameter (Dm.)
1000 meters = 1 kilometer (Km.)

### English to Metric

1 in. = 2.5400 cm.
1 ft. = 30.480 cm.
1 mi. = 1.6093 Km.

### Metric to English

1 cm. = 0.3937 in.
1 m. = 39.37 in. = 3.2808 ft.
1 Km. = 0.6214 mi.

## *B. Units of Area or Surface*

1 square yard = 9 square feet = 1296 square inches
1 acre (A.) = 160 square rods = 4840 square yards
1 square mile = 640 acres = 102400 square rods

## *C. Units of Measurement of Capacity*

### Dry Measure

2 pints (pt.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

### Liquid Measure

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
1 gallon = 231 cu. in.

## *D. Metric Units to English Units*

1 liter = 1000 cu. cm. = 61.02 cu. in. = 1.0567 liquid quarts
1 quart = .94636 liter = 946.36 cu. cm.
1000 grams = 1 kilogram (Kg.) = 2.2046 pounds (lb.)
1 pound = .453593 kilogram = 453.59 grams

## *E. Other Numbers*

$\pi$ = ratio of circumference to diameter of a circle
= 3.14159265
1 radian = angle subtended by an arc equal to the radius
$= 57^\circ\,17'\,44''.8 = 57^\circ.2957795 = 180^\circ/\pi$
1 degree = 0.01745329 radian, or $\pi/180$ radians
Weight of 1 cu. ft. of water = 62.425 lb.

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